NOTE ON AN ADJACENCY MATRIX OF A GRAPH $G$

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ABSTRACT. Let $G = (V(G), E(G))$ be a finite undirected graph with no loops or multiple edges with a vertex set $V(G)$ and an edge set $E(G)$. In this paper, we have considered the statement of the lemma in [4]. We have given some new results on the largest eigenvalue of an adjacency matrix of a graph $G$.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite undirected graph with no loops or multiple edges with a vertex set $V(G) = \{v_1, \ldots, v_n\}$ and an edge set $E(G)$ such that $|E(G)| = m$. We denote an edge which connects the vertex $v_i$ and $v_j$ by $e_{ij} = v_iv_j = (v_i, v_j)$. Let $A(G) = [a_{ij}]$ denote an adjacency matrix of a graph $G$. Let $\rho_1(G), \ldots, \rho_n(G)$ be the eigenvalues of $A(G)$ such that $\rho_n(G) \leq \cdots \leq \rho_1(G)$. The spectral radius, $\rho(G)$ of a graph $G$ corresponding to an adjacency matrix $A(G)$ is the largest eigenvalues of $A(G)$. Here $\rho(G) = \rho_1(G)$.

Let $L(G) = [l_{ij}]$ denote a Laplacian matrix of a graph $G$. Let $\lambda_1(G), \ldots, \lambda_n(G)$ be the eigenvalues of $L(G)$ such that $\lambda_n(G) \leq \cdots \leq \lambda_1(G)$. The spectral radius, $\lambda_1(G)$ of a graph $G$ corresponding to a Laplacian matrix $L(G)$ is the largest eigenvalues of $L(G)$.

If we compare the proof of Lemma 2.1 given in [4] with example, then we can observe that the proof of lemma 2.1 given in [4] need modification.

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In this paper, we have checked the lemma 2.1 given in [4] by using example. We have given some results on an eigenvalues of $A(G)$. For spectral radius, $\rho(G)$ of a graph $G$ (see [1–8].)

2. SOME RESULTS ON LARGEST EIGENVALUE OF AN ADJACENCY MATRIX OF A GRAPH $G$

In 1988, Hong Yuan shown that the spectral radius

\begin{equation}
\rho(G) \leq \sqrt{2m - n + 1}
\end{equation}

with equality if and only if $G$ is isomorphic to $K_{1,n-1}$ or $K_n$ (see [1]). From this results one can show that if $G$ is a simple planar graph with $n \geq 3$ vertices and $m$ edges then $\rho(G) \leq \sqrt{5m - 11}$. If $G$ is a connected graph with $n$ vertices then

$$
\sum_{i=2}^{n} \rho_i^2(G) \geq n - 1.
$$

We can also find the another upperbound of $\rho(G)$ given as follows:

**Lemma 2.1.** Let $G = (V(G), E(G))$ be a connected graph with $n$ vertices and $m$ edges. Let vertex set $V(G) = \{v_1, \cdots, v_n\}$. Let $d_i$ is the degree of vertex $v_i$. Let $\delta_1 = \min \{d_i : v_i \in V(G)\}$. Let $\delta_2 = \max \{d_i : v_i \in V(G)\}$. If $X = (x_1, \cdots, x_n)^t$ be an eigenvector of $A(G)$ corresponding to an eigenvalue $\rho(G)$ such that $||x|| = 1$ Then

$$
\rho(G) \leq \sqrt{2m - (n - 1) \sum_{i=1}^{n} x_{v_i}^2},
$$

where $x_{v_i} = \min \{|x_j| : (v_i, v_j) \notin E(G)\}$.

**Proof.** Let $G = (V(G), E(G))$ be a connected graph with $n$ vertices and $m$ edges. Let vertex set $V(G) = \{v_1, \cdots, v_n\}$. Let $d_i$ is the degree of vertex $v_i$. Let $\delta_1 = \min \{d_i : v_i \in V(G)\}$. Let $A(G)$ be an adjacency matrix of $G$. Let $A_i(G)$ denote $i^{th}$ row of $A(G)$ which is corresponding to a vertex $v_i$. Let $S_{v_i}(A(G))$ denotes $i^{th}$ row sum of $A(G)$. Therefore $S_{v_i}(A(G)) = d_i$. Let $X = (x_1, \cdots, x_n)^t$ be an eigenvector of $A(G)$ corresponding to an eigenvalue $\rho(G)$ such that $||x|| = 1$ i.e

$$
x_1^2 + \cdots + x_n^2 = 1.
$$

Therefore

\begin{equation}
A(G)X = \rho(G)X
\end{equation}
Here $x_i$ denote an eigencomponent of $X$ correspond to a vertex $v_i$. Let for the vertex $v_i$, $x_{v_i} = \min\{|x_j| : (v_i, v_j) \notin E(G)\}$.

Let $Y(v_i)$ denote the vector obtained from $X$ by replacing $x_j$ with 0 if $(v_i, v_j) \notin E(G)$. From $i^{th}$ equation of (2.2), we have

$$A_i(G)Y(v_i) = A_i(G)X = \rho(G)X = \rho(G)x_i.$$ 

Hence by the Cauchy-Schwartz inequality

$$\rho^2(G)x_i^2 = |A_i(G)Y(v_i)|^2 \leq |A_i(G)|^2 |Y(v_i)|^2. \tag{2.3}$$

We know that $|A_i(G)|^2 = \sum_{j=1}^{n} a_{ij}^2 = d_i$.

We know that $x_1^2 + \cdots + x_n^2 = 1$. Therefore

$$\sum_{(v_i, v_j) \in E(G)} x_j^2 + \sum_{(v_i, v_j) \notin E(G)} x_j^2 = 1$$

Then

$$\sum_{(v_i, v_j) \in E(G)} x_j^2 = 1 - \sum_{(v_i, v_j) \notin E(G)} x_j^2$$

$$|Y(v_i)|^2 = \sum_{(v_i, v_j) \in E(G)} x_j^2 = 1 - \sum_{(v_i, v_j) \notin E(G)} x_j^2.$$ 

From equation (2.3), we have

$$\rho^2(G)x_i^2 \leq d_i(1 - \sum_{(v_i, v_j) \notin E(G)} x_j^2).$$

Taking sum on both side over all vertices of a graph $G$, we have

$$\sum_{i=1}^{n} \rho^2(G)x_i^2 \leq \sum_{i=1}^{n} (d_i(1 - \sum_{(v_i, v_j) \notin E(G)} x_j^2)).$$

$$\rho^2(G)\sum_{i=1}^{n} x_i^2 \leq 2m - (\sum_{i=1}^{n} d_i \sum_{(v_i, v_j) \notin E(G)} x_j^2)).$$

$$\rho^2(G) \leq 2m - (\sum_{i=1}^{n} d_i \sum_{(v_i, v_j) \notin E(G)} x_j^2)). \tag{2.4}$$
Let
\[ \sum_{i=1}^{n} d_i \left( \sum_{(v_i, v_j) \notin E(G)} x_j^2 \right) = \sum_{i=1}^{n} d_i \left( x_i^2 + \sum_{(v_i, v_j) \notin E(G), i \neq j} x_j^2 \right) \]
\[ = \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} \left( d_i \sum_{(v_i, v_j) \notin E(G), i \neq j} x_j^2 \right) \]
\[ \geq \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} \left( \sum_{(v_i, v_j) \notin E(G), i \neq j} x_j^2 \right) \]
\[ \geq \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} \left( \sum_{(v_i, v_j) \notin E(G), i \neq j} x_{v_i}^2 \right) \]
\[ \geq \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n} \left( (n - (d_i + 1)) x_{v_i}^2 \right) \]
\[ \geq \sum_{i=1}^{n} \left[ d_i + (n - (d_i + 1)) \right] x_{v_i}^2 \]
\[ \geq (n - 1) \sum_{i=1}^{n} x_{v_i}^2. \]

Therefore
\[ \sum_{i=1}^{n} d_i \left( \sum_{(v_i, v_j) \notin E(G)} x_j^2 \right) \geq (n - 1) \sum_{i=1}^{n} x_{v_i}^2, \]

(2.5)
\[ - \sum_{i=1}^{n} d_i \left( \sum_{(v_i, v_j) \notin E(G)} x_j^2 \right) \leq -(n - 1) \sum_{i=1}^{n} x_{v_i}^2. \]

Put (2.5) in (2.4), we get
\[ \rho^2(G) \leq 2m - (n - 1) \sum_{i=1}^{n} x_{v_i}^2, \]
\[ \rho^2(G) \leq 2m - (n - 1) \sum_{i=1}^{n} x_{v_i}^2, \]

where \( x_{v_i} = \min \{|x_j| : (v_i, v_j) \notin E(G)\} \).
Therefore,

$$\rho(G) \leq \sqrt{2m - (n - 1) \sum_{i=1}^{n} x_{vi}^2}.$$ 

Remark 2.1. If \( \sum_{i=1}^{n} x_{vi}^2 = 1 \), we get the equation (2.1), one can observe that

$$\rho(G) \leq \sqrt{2m - n + 1} \leq \sqrt{2m - (n - 1) \sum_{i=1}^{n} x_{vi}^2}.$$ 

Lemma 2.2. Let \( G = (V(G), E(G)) \) be a connected graph with \( n \) vertices and \( m \) edges. Let vertex set \( V(G) = \{v_1, \cdots, v_n\} \). Let \( d_i \) is the degree of vertex \( v_i \). Let \( \delta_1 = \min\{d_i : v_i \in V(G)\} \). Let \( \delta_2 = \max\{d_i : v_i \in V(G)\} \). Then

$$\rho(G) \leq \sqrt{\delta_2 \sum_{i=1}^{n} \left( \sum_{(v_i,v_j) \in E(G)} x_{ij}^2 \right)}.$$ 

Proof. Let \( G = (V(G), E(G)) \) be a connected simple graph with \( n \) vertices and \( m \) edges. Let vertex set \( V(G) = \{v_1, \cdots, v_n\} \). Let \( d_i \) is the degree of vertex \( v_i \). Let \( \delta_1 = \min\{d_i : v_i \in V(G)\} \). Let \( A_i(G) \) denote \( i^{th} \) row of \( A(G) \) which is corresponding to a vertex \( v_i \). Let \( S_{vi}(A(G)) \) denotes \( i^{th} \) rowsum of \( A(G) \). Therefore \( S_{vi}(A(G)) = d_i \). Let \( X = (x_1, \cdots, x_n)^t \) be an eigenvector of \( A(G) \) corresponding to an eigenvalue \( \rho(G) \) such that \( ||x|| = 1 \) i.e

$$x_1^2 + \cdots + x_n^2 = 1.$$ 

Therefore

(2.6) \[ A(G)X = \rho(G)X. \]

Here \( x_i \) denote an eigencomponent of \( X \) correspond to a vertex \( v_i \). Let \( Y(v_i) \) denote the vector obtained from \( X \) by replacing \( x_j \) with 0 if \( (v_i, v_j) \notin E(G) \). From \( i^{th} \) equation of (2.6), we have

$$A_i(G)Y(v_i) = A_i(G)X = \rho(G)X = \rho(G)x_i.$$

Hence by the Cauchy-Schwartz inequality

(2.7) \[ \rho^2(G)x_i^2 = |A_i(G)Y(v_i)|^2 \leq |A_i(G)|^2||Y(v_i)||^2. \]
We know that $|A_i(G)|^2 = \sum_{j=1}^{n} a_{ij}^2 = d_i$. We know that $x_1^2 + \cdots + x_n^2 = 1$. Therefore
\[
\sum_{(v_i,v_j)\in E(G)} x_j^2 + \sum_{(v_i,v_j)\notin E(G)} x_j^2 = 1.
\]
\[
\sum_{(v_i,v_j)\in E(G)} x_j^2 = 1 - \sum_{(v_i,v_j)\notin E(G)} x_j^2.
\]
\[
|Y(v_i)|^2 = \sum_{(v_i,v_j)\in E(G)} x_j^2 = 1 - \sum_{(v_i,v_j)\notin E(G)} x_j^2.
\]
From equation (2.7), we have
\[
\rho^2(G)x_i^2 \leq d_i(1 - \sum_{(v_i,v_j)\notin E(G)} x_j^2).
\]
Taking sum on both side over all vertices of a graph $G$, we have
\[
\sum_{i=1}^{n} \rho^2(G)x_i^2 \leq \sum_{i=1}^{n} (d_i(1 - \sum_{(v_i,v_j)\notin E(G)} x_j^2)).
\]
\[
\rho^2(G)(\sum_{i=1}^{n} x_i^2) \leq 2m - (\sum_{i=1}^{n} d_i \sum_{(v_i,v_j)\notin E(G)} x_j^2)).
\]
\[
\rho^2(G) \leq 2m - (\sum_{i=1}^{n} d_i \sum_{(v_i,v_j)\notin E(G)} x_j^2)).
\]
We know that
\[
\sum_{i=1}^{n} (d_i \sum_{j=1}^{n} x_j^2) = \sum_{i=1}^{n} (d_i \sum_{(v_i,v_j)\in E(G)} x_j^2) + \sum_{i=1}^{n} (d_i \sum_{(v_i,v_j)\notin E(G)} x_j^2).
\]
\[
\sum_{i=1}^{n} (d_i \sum_{(v_i,v_j)\notin E(G)} x_j^2) = \sum_{i=1}^{n} (d_i \sum_{j=1}^{n} x_j^2) - \sum_{i=1}^{n} (d_i \sum_{(v_i,v_j)\in E(G)} x_j^2).
\]
\[
\geq \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} (\delta_2 \sum_{(v_i,v_j)\in E(G)} x_j^2)
\]
\[
\geq 2m - \sum_{i=1}^{n} (\delta_2 \sum_{(v_i,v_j)\in E(G)} x_j^2)
\]
(2.8) \[ -\sum_{i=1}^{n} (d_i \sum_{(v_i,v_j)\notin E(G)} x_j^2) \leq -(2m - \sum_{i=1}^{n} (\delta_2 \sum_{(v_i,v_j)\in E(G)} x_j^2)) \]
From (2.4) and (2.8), we have
\[
\rho^2(G) \leq 2m - \left(2m - \sum_{i=1}^{n} \left(\sum_{\delta_2(v_i, v_j) \in E(G)} x_i^2 \right) \right)
\]
\[
\leq \delta_2 \sum_{i=1}^{n} \left(\sum_{\delta_2(v_i, v_j) \in E(G)} x_i^2 \right).
\]
Hence
\[
\rho(G) \leq \sqrt{\delta_2 \sum_{i=1}^{n} \left(\sum_{\delta_2(v_i, v_j) \in E(G)} x_i^2 \right)}.
\]

In 1997, Hong Yuan shown that if \( G \) be a simple graph with \( n \) vertices and \( X = (x_1, \ldots, x_n)^t \) be an eigenvector corresponding to an eigenvalue \( \rho(G) \) such that \( ||x|| = 1 \) then \( \rho(G) \leq \sum_{i=1}^{n} d_i x_i^2 \), see [2].

3. Note on an adjacency matrix

In this section, we have considered the statement of the lemma 2.1 and its proof which is given in [4] for completion of this article.

**Lemma 3.1.** [4] Let \( G \) be a connected graph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and \( d_i \) be the degree of vertex \( v_i, i = 1, \ldots, n \). Then
\[
(3.1) \quad \rho(G) \leq \max_{(v_i, v_j) \in E(G)} \sqrt{d_i d_j},
\]
where \( E(G) \) is the edge set of \( G \). Moreover, the equality in (3.1) holds if and only if \( G \) is a regular or bipartite semiregular graph.

**Remark 3.1.** In [4], it is given in the proof that \( X \) be a Perron vector of \( G \), where \( x_i \) corresponds to the vertex \( v_i \). Let \( x_s = \max_{v_i \in V(G)} x_i \) and \( x_t = \max_{(v_i, v_s) \in E(G)} x_i \). From \( A(G)X = \rho(G)X \), we have
\[
(3.2) \quad \rho(G)x_s = \sum_{v_i \in N_G(v_s)} x_i \leq \sum_{v_i \in N_G(v_s)} x_t = d_s x_t,
\]
\[
(3.3) \quad \rho(G)x_t = \sum_{v_i \in N_G(v_t)} x_i \leq \sum_{v_i \in N_G(v_t)} x_t = d_t x_s,
\]
where $N_G(S)$ denotes the neighbors in $G$ of $S$. Hence
\[
\rho(G)^2 x_s x_t \leq d_s d_t x_s x_t \\
\rho(G) \leq \sqrt{d_s d_t}.
\]

3.1. **Verification proof by an example.** In this subsection, we claim that equation (3.2) and (3.3) not correct step in general. The proof of this statement is given as follows:

**Proof.** Suppose if possible equation (3.2) is correct.

Let $G = (V(G), E(G))$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_{10}\}$ and $E(G) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_5, v_6), (v_7, v_6), (v_7, v_8), (v_7, v_9), (v_7, v_{10})\}$. The adjacency matrix $A(G)$ of $G$ is given by

\[
A(G) = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

The largest eigenvalue of the matrix $A(G)$ is $\rho(G) = 2.1987$. An eigenvector $X$ corresponding to an eigenvalue $\rho(G)$ is given by

\[
X = \begin{bmatrix}
-0.0000 & -0.2988 & -0.2988 & -0.2988 & -0.5481 & -0.0000 & -0.6570 & 0 & 0 & 0
\end{bmatrix}.
\]

Let $x_s = \max_{v_i \in V(G)} x_i = 0$ (which is corresponding to vertices $v_8, v_9, v_{10}$). In $G$, we have $(v_7, v_8), (v_7, v_9), (v_7, v_{10}) \in E(G)$. Hence $x_t = \max_{(v_i, v_j) \in E(G)} x_i = -0.6570$ (which is corresponding to a vertex $v_7$). From equation (3.2), we have

\[
(3.4) \quad \rho(G)x_s = \sum_{v_i \in N_G(v_s)} x_i \leq \sum_{v_i \in N_G(v_s)} x_t = d_s x_t.
\]

Put the value of $x_s, x_t, \rho(G), d_s$ in equation (3.4), we get

\[
(2.1987)0 = \sum_{v_i \in N_G(v_s)} x_i \leq \sum_{v_i \in N_G(v_s)} x_t = x_t = d_s x_t = (1)(-0.6570).
\]
Therefore \(0 \leq -0.6570\), which is not possible. Hence the equation (3.2), given in proof of lemma 2.1 in [4] do not hold in general. \(\Box\)

### 3.2. New result.

**Lemma 3.2.** [3] Let \(G\) be a connected graph with vertex set \(V(G) = \{v_1, \cdots, v_n\}\) such that \(|V(G)| = n\). Let \(A(G)\) be an adjacency matrix of \(G\). Let \(P\) be any polynomial and \(S_{v_i}(P(A(G)))\) be a rowsums of \(P(A(G))\) corresponding to the vertex \(v_i\). Then

\[
\min S_{v_i}(P(A(G))) \leq P(\rho(G)) \leq \max S_{v_i}(P(A(G))).
\]

Moreover, equality holds if and only if the rowsums of \(P(A(G))\) are all equal.

**Lemma 3.3.** Let \(G\) be a connected graph with vertex set \(V(G) = \{v_1, \cdots, v_n\}\) and \(d_i\) be the degree of vertex \(v_i\), \(i = 1, \cdots, n\). Let \(\rho(G)\) be a largest eigenvalue of an adjacency matrix \(A(G)\). Let \(x = (x_1, \cdots, x_n)^t\) be an eigenvector corresponding to an eigenvalue \(\rho(G)\). Then

\[
\rho(G) \leq \max_{(v_i, v_j) \in E(G)} \sqrt{d_i d_j},
\]

where \(E(G)\) is the edge set of \(G\).

**Proof.** Let \(G\) be a connected graph with vertex set \(V(G) = \{v_1, \cdots, v_n\}\) and \(d_i\) be the degree of vertex \(v_i\), \(i = 1, \cdots, n\). Let \(\rho(G)\) be a largest eigenvalue of an adjacency matrix \(A(G)\). Let \(x = (x_1, \cdots, x_n)^t\) be an eigenvector corresponding to an eigenvalue \(\rho(G)\). Let \(N_i = \{v_\alpha : (v_i, v_\alpha) \in E(G)\}\).

Let \(S_{v_i}(A^k(G))\) is the number of walks of length \(k\) in \(G\) which begin at \(v_i\). Therefore \(S_{v_i}(A(G))\) is \(d_i\).

\[
S_{v_i}(A^2(G)) = d_i + \sum |N_i \cap N_j|
\]

\[
S_{v_i}(A^2(G)) = \sum_{(v_i, v_j) \in E(G)} d_j.
\]

Let \(d_k = \max\{d_j : (v_i, v_j) \in E(G)\}\). Therefore we have

\[
S_{v_i}(A^2(G)) = \sum_{(v_i, v_j) \in E(G)} d_j \leq \sum_{(v_i, v_j) \in E(G)} d_k = d_i d_k.
\]
By using lemma 3.2, we have

\[ \rho^2(G) \leq \max_{v_i \in V(G)} S_{v_i}(A^2(G)) \]

\[ \rho^2(G) \leq \max \{ d_i d_j : (v_i, v_j) \in E(G) \} \]

\[ \rho(G) \leq \max_{(v_i, v_j) \in E(G)} \sqrt{d_i d_j}. \]

\[ \square \]

**Lemma 3.4.** Let \( G \) be a connected graph with vertex set \( V(G) = \{v_1, \cdots, v_n\} \) and \( d_i \) be the degree of vertex \( v_i, i = 1, \cdots, n \). Let \( \rho(G) \) be a largest eigenvalue of an adjacency matrix \( A(G) \). Let \( x = (x_1, \cdots, x_n)^t \) be an eigenvector corresponding to an eigenvalue \( \rho(G) \). Let for a vertex \( v_i \in V(G) \), \( d_k = \max \{ d_j : (v_i, v_j) \in E(G) \} \) and \( d_\alpha = \max \{ d_j : (v_i, v_j) \in E(G) \} \) be second maximum degree among all degree of vertices adjacent to \( v_i \), hence \( d_\alpha \leq d_k \)

Then

\[ \rho(G) \leq \max_{(v_i, v_\alpha) \in E(G), (v_i, v_k) \in E(G)} \sqrt{(d_i - 1)d_\alpha + d_k}, \]

where \( E(G) \) is the edge set of \( G \).

**Proof.** Let \( G \) be a connected graph with vertex set \( V(G) = \{v_1, \cdots, v_n\} \) and \( d_i \) be the degree of vertex \( v_i, i = 1, \cdots, n \). Let \( \rho(G) \) be a largest eigenvalue of an adjacency matrix \( A(G) \). Let \( x = (x_1, \cdots, x_n)^t \) be an eigenvector corresponding to an eigenvalue \( \rho(G) \). Let \( N_i = \{v_\alpha : (v_i, v_\alpha) \in E(G)\} \).

Let \( S_{v_i}(A^k(G)) \) is the number of walks of length \( k \) in \( G \) which begin at \( v_i \). Therefore \( S_{v_i}(A(G)) \) is \( d_i \).

\[ S_{v_i}(A^2(G)) = d_i + \sum |N_i \cap N_j|. \]

\[ S_{v_i}(A^2(G)) = \sum_{(v_i, v_j) \in E(G)} d_j. \]

Let \( d_k = \max \{ d_j : (v_i, v_j) \in E(G) \} \). Therefore we have

\[ S_{v_i}(A^2(G)) = \sum_{(v_i, v_j) \in E(G)} d_j = ( \sum_{(v_i, v_j) \in E(G)} d_j ) + d_k \]

\[ \leq ( \sum_{(v_i, v_j) \in E(G)} d_\alpha ) + d_k = (d_i - 1)d_\alpha + d_k. \]
By using lemma 3.2, we have

\[ \rho^2(G) \leq \max_{v_i \in V(G)} S_{v_i}(A^2(G)) \]

\[ \rho^2(G) \leq \max \{ (d_i - 1)d_\alpha + d_k : (v_i, v_j) \in E(G) \} \]

\[ \rho(G) \leq \max_{(v_i, v_\alpha) \in E(G), (v_i, v_k) \in E(G)} \sqrt{(d_i - 1)d_\alpha + d_k} . \]

\[ \Box \]

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