SOME FIXED POINT RESULTS FOR $W-$DISTANCE IN METRIC SPACE

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ABSTRACT. The notion of $\alpha - \psi$-contractive mapping for $w-$distance came into existence of a long period of mathematical analysis and investigation due to Lakzian. In this paper, we introduce the $\alpha - \psi$-contractive multifunction for $w-$distance and give a fixed point result for these multifunction. We also obtain a fixed point result for self-map in complete metric space satisfying contractive conditions.

1. INTRODUCTION

In 1922, Banach [1] introduced the concept of contraction principle. Kada et al. [3] introduced the concept of $w$-distance for metric space. Using the concept of $w$-distance, Suzuki and Takahashi [10] established fixed point results for multivalued function. Samet et al. [9] introduced the concept of $\alpha - \psi$-contractive and $\alpha$-admissible mappings and established fixed point theorems for $\alpha - \psi$-contractive mapping satisfying $\alpha$-admissible condition for complete metric space. Hasanzade Asl et al. [2] introduced $\alpha_* - \psi$-contractive multifunction. They also obtained fixed point results for self maps in complete metric space satisfying contractive conditions. More details about multifunction are given in [4], [6-9] and [11].

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$. Also, we denote the family of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi$ satisfying the

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following conditions: $\psi$ is non-decreasing and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n$ is the $n$th iterate of $\psi$, see [9]. Let $(X, d)$ be a metric space and let $T : X \to X$ be a given mapping. We say that $T$ is an $\alpha - \psi$-contractive mapping if there exist two function $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, for all $x, y \in X$, see [9]. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two given mappings. Then $T$ is called an $\alpha$-admissible mapping if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$, see [3]. Let $X$ be a metric space endowed with a metric $d$. We introduce the concept of $\alpha$-admissible self-map in complete metric spaces satisfying contractive conditions. We have also obtain a fixed point theorem for multifunction. We have also obtained a fixed point for multifunction. We say that $T = \psi$ for any $x, y, z \in X$. In this case, we say that $\psi$ is Hausdorff generalized metric, $\alpha_a(A, B) = \inf(\alpha(a, b) : a \in A, b \in B)$ and $2^X$ denote the family of all non-empty subsets of $X$. Also, we say that $T$ is $\alpha$-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha_a(Tx, Ty) \geq 1$.

Motivated by the results of Lakzian et al. [5] and Hasanzade Asl et al. [2], we introduce the concept of $\alpha - \psi$-contractive multifunction for $w$-distance and give a fixed point theorem for multifunction. We have also obtained a fixed point theorem for self-map in complete metric spaces satisfying contractive conditions.
2. Main Result

Definition 2.1. Let \((X,d)\) be a metric space with \(w\)-distance \(p\). Let \(T: X \to 2^X\) be a closed-valued multifunction, \(\psi \in \Psi\) and \(\alpha: X \times X \to [0,\infty)\) be given mappings. We say that \(T\) is \((\alpha_*, \psi, p)\) contractive multifunction whenever

\[
\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(p(x, y))
\]

for \(x, y \in X\) and where \(H\) is Hausdorff generalized metric \(\alpha_*(A, B) = \inf\{\alpha(a, b): a \in A, b \in B\}\) and \(2^X\) denotes the family of all non-empty subsets of \(X\). We say that \(T\) is \(\alpha_*\)-admissible whenever \(\alpha(x, y) \geq 1\) implies \(\alpha_*(Tx, Ty) \geq 1\).

Theorem 2.1. Let \(p\) be a \(w\)-distance on a complete metric space \((X,d)\), \(\alpha: X \times X \to [0,\infty)\) be a function and \(\psi \in \Psi\) be strictly increasing function. Let \(T: X \to 2^X\) be a closed-valued \((\alpha_*, \psi, p)\) contractive multifunction on \(X\). Suppose that the following condition hold:

(i) \(T\) is \(\alpha_*\)-admissible;
(ii) there exists \(x_0 \in X\) and \(x_1 \in Tx_0\) such that \(\alpha(x_0, x_1) \geq 1\);
(iii) for any sequence \(\{x_n\}\) in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to \infty\) then \(\alpha(x_n, x) \geq 1\) for all \(n\).

Then \(T\) has a fixed point.

Proof. From (ii), if \(x_1 = x_0\), then we have nothing to prove. Let \(x_1 \neq x_0\). If \(x_1 \in Tx_1\), then \(x_1\) is a fixed point of \(T\). Let \(x_1 \neq Tx_1\) and \(q > 1\) be given. Then

\[
0 < p(x_1, Tx_2) \leq \alpha_*(Tx_0, Tx_1)H(Tx_0, Tx_1) < q\alpha_*(Tx_0, Tx_1)H(Tx_0, Tx_1).
\]

Hence, there exists \(x_2 \in Tx_1\) such that

\[
0 < p(x_1, x_2) \leq q\alpha_*(Tx_0, Tx_1)H(Tx_0, Tx_1) \leq q\psi(p(x_0, x_1)).
\]

It is clear that \(x_2 \neq x_1\) and \(\alpha(x_1, x_2) \geq 1\). Thus, \(\alpha_*(Tx_1, Tx_2) \geq 1\). Now put \(t_0 = p(x_0, x_1)\). Then, \(t_0 > 1\) and \(p(x_1, x_2) < q\psi(t_0)\). Since \(\psi\) is strictly increasing, \(\psi(p(x_1, x_2)) < \psi(q\psi(t_0))\). Put \(q_1 = \frac{\psi(p(x_1, x_2))}{\psi(q\psi(t_0))}\). Then \(q_1 > 1\). If \(x_2 \in Tx_2\), then \(x_2\) is a fixed point of \(T\). Assume that \(x_2 \neq Tx_2\). Then

\[
0 < p(x_2, Tx_2) \leq \alpha_*(Tx_1, Tx_2)H(Tx_1, Tx_2) < q_1\alpha_*(Tx_1, Tx_2)H(Tx_1, Tx_2).
\]

Hence, there exists \(x_3 \in Tx_3\) such that

\[
0 < p(x_2, x_3) \leq q_1\alpha_*(Tx_1, Tx_2)H(Tx_1, Tx_2) \leq q_1\psi(p(x_1, x_2)) = \psi(q\psi(t_0)).
\]
It is clear that \( x_3 \neq x_2, \alpha(x_2, x_3) \geq 1 \) and \( \psi(p(x_2, x_3)) < \psi^2(\psi(t_0)) \). Now, put \( q_2 = \psi^2(\psi(t_0)) \). Then \( q_2 > 1 \). If \( x_3 \in Tx_3 \), then \( x_3 \) is a fixed point of \( T \). Assume that \( x_3 \neq Tx_3 \). Then

\[
0 < p(x_3, Tx_3) \leq \alpha_s(Tx_2, Tx_3)H(Tx_2, Tx_3) < q_2 \alpha_s(Tx_2, Tx_3)H(Tx_2, Tx_3).
\]

Thus, there exists \( x_4 \in Tx_3 \) such that

\[
0 < p(x_3, x_4) < q_2 \alpha_s(Tx_2, Tx_3)H(Tx_2, Tx_3) \leq q_2 \psi(p(x_2, x_3)) = \psi^2(\psi(t_0)).
\]

Since \( T \) is an \( \alpha_s \)-admissible mapping, we have \( \alpha(x_0, x_1) \geq 1 \) implies \( \alpha_s(Tx_0, Tx_0) = \alpha(x_1, x_2) \geq 1 \). Using by mathematical induction it follows that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \).

By continuing this process, we obtain a sequence \( \{x_n\} \) in \( X \) such that \( x_n \in Tx_{n-1}, x_n \neq x_{n-1}, \alpha(x_n, x_{n+1}) \geq 1 \) and \( p(x_n, x_{n+1}) \leq \psi^{n-1}(\psi(t_0)) \) for all \( n \). Now, for each \( m > n \), we have

\[
p(x_n, x_m) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_m-1, x_m) \\
\leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(\psi(t_0)).
\]

Hence, \( x_n \) is a Cauchy sequence in \( X \). Choose \( x^* \in X \) such that \( x_n \to x^* \). Since \( \alpha(x_n, x^*) \geq 1 \) for all \( n \) and \( T \) is \( \alpha_s \)-admissible, \( \alpha_s(Tx_n, Tx^*) \geq 1 \) for all \( n \), thus

\[
p(x^*, Tx^*) \leq p(Tx^*, Tx_n) + p(Tx_n, x^*) \\
\leq \alpha_s(Tx_n, Tx^*)H(Tx_n, Tx^*) + p(x_{n+1}, x^*) \\
\leq \psi(p(x_n, x^*)) + p(x_{n+1}, x^*) \to 0
\]

as \( n \to \infty \). Therefore \( p(x^*, Tx^*) = 0 \) and so \( x^* \in Tx^* \). Then, \( x^* \) is a fixed point of \( T \). \( \square \)

**Example 1.** Consider \( X = [0, \infty) \) which is indeed a complete metric space under the usual metric \( d(x, y) = |x - y| \) for all \( x, y \in X \) where in by defining \( p(x, y) = y \), we have obtain that \( p \) is a \( w \)-distance on \( (X, d) \). Let \( T : X \to 2^X \) be given

\[
Tx = \begin{cases} 
  x^2 - \frac{5}{2}, & \text{for all } x > 1, \\
  \frac{x}{3}, & \text{for all } x \in [0, 1] 
\end{cases}
\]
and $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0, 1], \\
0, & \text{otherwise}
\end{cases}$$

and $\alpha(x, y) = 0$ whenever $x \notin [0, 1]$ or $y \notin [0, 1]$.

Then it is easy to check that $T$ is an $\alpha$–admissible and $(\alpha, \psi, p)$–contractive multifunction, where $\psi(t) = \frac{t}{3}$ for all $t \geq 0$. Put $x_0 = \frac{1}{3}, x_1 = 1$ then $\alpha(x_0, x_1) \geq 1$.

Also, if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n$. Note that $T$ has infinitely many fixed points.

**Theorem 2.2.** Let $p$ be a $w$–distance on a complete metric space $(X, d)$, $\alpha : X \times X \to [0, \infty)$ be a function, $\psi \in \Psi$ and $T : X \to X$ be a self map on $X$ such that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(m(x, y))$$

for all $x, y \in X$, where

$$m(x, y) = \max\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

Suppose that the following conditions hold:

(i) $T$ is $\alpha$–admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) for any sequence $\{x_n\}$ in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all $n$.

Then $T$ has a fixed point.

**Proof.** From (ii), we take $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define the sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some $n$, then $x^* = x_n$ is a fixed point of $T$. Assume that $x_n \neq x_{n+1}$ for all $n$. Since, $T$ is $\alpha$–admissible then

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$$

using by mathematical induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1$$
for all natural number \( n \). Thus, for each natural number \( n \), we have

\[
p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)p(Tx_{n-1}, Tx_n)
\]

\[
\leq \psi(\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n) + p(x_{n-1}, Tx_n), p(x_n, x_{n-1})/2\})
\]

\[
\leq \psi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})/2, p(x_n, x_{n-1})/2\})
\]

\[
\leq \psi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})/2, p(x_n, x_{n-1})/2\})
\]

\[
\leq \psi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})/2, p(x_n, x_{n-1})/2\})
\]

\[
\leq \psi(p(x_n, x_{n-1}))
\]

using by mathematical induction

\[
p(x_n, x_{n+1}) \leq \psi^n(p(x_1, x_0))
\]

for all \( n \in \mathbb{N} \).

According to the definition of \( \psi \), we conclude that \( \lim_{n \to \infty} \psi^n(p(x_1, x_0)) = 0 \). This implies that

\[
\lim_{n \to \infty} p(x_n, x_{n+1}) = 0.
\]

Next, we will prove that \( \{x_n\} \) is a Cauchy sequence. Suppose that \( m, n \in \mathbb{N} \) with \( m > n \). Then we have

\[
p(x_n, x_m) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m)
\]

\[
\leq \sum_{i=n}^{m-1} \psi^i(p(x_1, x_0))
\]

\[
\leq \sum_{i=n}^{\infty} \psi^i(p(x_1, x_0)) \to 0
\]

as \( n \to \infty \). Hence the sequence \( \{x_n\} \) is a Cauchy sequence in \((X, d)\). Thus, there exists \( x^* \in X \) such that \( x_n \to x^* \). By using assumption, we have \( \alpha(x_n, x^*) \geq 1 \).
for all \(n\). Thus,
\[
p(Tx^*, x^*) \leq p(Tx^*, Tx_n) + p(Tx_n, x^*) \\
\leq \alpha(x_n, x^*)p(Tx^*, Tx_n) + p(x_{n+1}, x^*) \\
\leq \psi(\max\{p(x_n, x^*), \frac{p(x_n, Tx_n) + p(x_n, Tx_{n+1})}{2}\}) \\
\leq \psi(\max\{p(x_n, x^*), \frac{p(x_n, x_{n+1}) + p(x^*, x_{n+1})}{2}\}) \\
\leq \psi(\max\{p(x_n, x^*), \frac{p(x_n, x^*) + p(x_n, x_{n+1}) + p(x_n, x_{n+1})}{2}\}) \\
\leq \psi(\max\{p(x^*, x^*), \frac{p(x^*, x^*) + p(x^*, x_{n+1}) + p(x_n, x_{n+1})}{2}\}) \\
\leq \psi(\max\{p(x^*, x^*), \frac{p(x^*, x^*) + p(x^*, x_{n+1}) + p(x^*, x_{n+1})}{2}\}) \\
\leq \psi(p(x^*, x^*) + p(x_{n+1}, x^*) + p(x_{n+1}, x^*))
\]
for sufficiently large \(n\). Hence \(p(x^*, Tx^*) = 0\) and so \(Tx^* = x^*\) i.e. \(x^*\) is a fixed point of \(T\).

**Example 2.** Consider \(X = [0, \infty)\) which is indeed a complete metric space under the usual metric \(d(x, y) = |x - y|\) for all \(x, y \in X\), defining \(p(x, y) = y\), we obtain that \(p\) is a \(w\)-distance on \((X,d)\). Let \(T : X \to X\) be given
\[
Tx = \begin{cases} 
3x - \frac{7}{3}, & \text{for all } x > 1, \\
\frac{x}{4}, & \text{for all } x \in [0,1] 
\end{cases}
\]
and \(\alpha : X \times X \to [0, \infty)\) as
\[
\alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0,1], \\
0, & \text{otherwise}
\end{cases}
\]
and \(\alpha(x, y) = 0\) whenever \(x \notin [0,1] \) or \(y \notin [0,1]\).

Then it is easy to check that \(T\) is an \(\alpha\)-admissible and \(\alpha(x, y)p(Tx, Ty) \leq \psi(m(x, y))\) for all \(x, y \in X\), and \(\psi(t) = \frac{t}{4}\) for all \(t \geq 0\). If \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n \in \mathbb{N}\). Then \(T\) has a fixed point.
3. CONCLUSION

Concluding we can say that our results are novel and improved results while concerning fixed point theorems in the metric space with w-distance. The $\alpha-\psi$-Contractive mapping defined here is quiet different one and relevant example supports the results very well.

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