FUZZY $\alpha$-TRANSLATIONS AND FUZZY $\beta$-MULTIPLICATIONS OF Z-ALGEBRAS

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ABSTRACT. In this article, fuzzy $\alpha$-translations, fuzzy extensions and fuzzy $\beta$-multiplications of fuzzy Z-Subalgebras (fuzzy Z-ideals) of Z-algebras are initiated and some interesting results are proved.

1. INTRODUCTION

A new class of algebra that arise from the propositional calculi is the Z-algebra introduced by Chandramouleeswaran et al. [1] in the year 2017. This algebra differs from the BCK, BCI, BF-algebras [2–5] and so on.

Zadeh [9] in the year 1965, introduced the notion of fuzzy sets as the generalization of set theory to deal with the problems of uncertainty under real physical world. Since then many authors fuzzified different algebraic structures. The idea of fuzzy translations and fuzzy multiplications have been discussed by Lee et al. [6]. Similar concept have been discussed in BF-algebras by Chandramouleeswaran et al. [2]. In [7, 8] we have launched the notion of fuzzy Z-Subalgebras and fuzzy Z-ideals respectively. In this paper, we examine fuzzy $\alpha$-translations and fuzzy $\beta$-multiplications of fuzzy Z-subalgebras and fuzzy Z-ideals in Z-algebras.

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2. Preliminaries

Now we collect the necessary definitions from the articles ([1], [7], [8], [9]).

**Definition 2.1.** [1] A Z-algebra (Z-algr) \((J, *, 0)\) is a nonempty set \(J\) with constant 0 and a binary operation \(*\) satisfying the following conditions:

1. \((Z1)\) \(u \ast 0 = 0\)
2. \((Z2)\) \(0 \ast u = u\)
3. \((Z3)\) \(u \ast u = u\)
4. \((Z4)\) \(u \ast \omega = \omega \ast u\) when \(u \neq 0\) and \(\omega \neq 0\) \(\forall u, \omega \in J\).

**Definition 2.2.** [9] A fuzzy set (fy set) \(A\) in a set \(J\) is defined by a membership function (msfn) \(\mu_A: J \rightarrow [0, 1]\).

**Definition 2.3.** [7] Let \((J, *, 0)\) be a Z-algr. A fy set \(A\) in \(J\) with msfn \(\mu_A\) is said to be a fy Z-Subalgebra (fy Z-Salgr) of a Z-algr \(J\) if \(\mu_A(u \ast \omega) \geq \min\{\mu_A(u), \mu_A(\omega)\}\) \(\forall u, \omega \in J\).

**Definition 2.4.** [8] Let \((J, *, 0)\) be a Z-algr. A fy set \(A\) in \(J\) with msfn \(\mu_A\) is said to be a fy Z-ideal (fy Z-idl) of a Z-algr \(J\) if:

1. \((i)\) \(\mu_A(0) \geq \mu_A(u)\);
2. \((ii)\) \(\mu_A(u) \geq \min\{\mu_A(u \ast \omega), \mu_A(\omega)\}\) \(\forall u, \omega \in J\).

3. Fuzzy \(\alpha\)-Translations and Fuzzy \(\beta\)-Multiplications of Fuzzy Z-Subalgebras (Fuzzy Z-ideals)

Hereafter, \((J, *, 0)\) denotes a Z-algr; and \(1 - \sup\{\mu_A(u) | u \in J\}\) is denoted by \(T\).

**Definition 3.1.** Let \(A\) be a fy set of a Z-algr \(J\) and let \(\alpha \in [0, T]\). A fy \(\alpha\)-translation (fy \(\alpha\)-tlt) \(A^T_\alpha\) of \(A\) with msfn \(\mu_{A^T_\alpha}: J \rightarrow [0, 1]\) is defined by \(\mu_{A^T_\alpha}(u) = \mu_A(u) + \alpha\), \(\forall u \in J\).
Example 1.

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From Table 1, \( J = \{0, s, p, g\} \) is a Z-algr. If a fy set \( A \) of \( J \) is given in Table 2, then \( A^T_{0.1} \) is a fy 0.1- tlt of \( A \).

**Theorem 3.1.** Let \( A \) be a fy set of Z-algr \( J \) and \( \alpha \in [0, T] \). Then the fy \( \alpha \)- tlt \( A^T_{\alpha} \) of \( A \) is a fy Z-Salgr of \( J \) \( \iff \) \( A \) is a fy Z-Salgr of \( J \).

**Definition 3.2.** When \( A_1 \) and \( A_2 \) are fy sets of Z-algr \( J \), \( A_2 \) is called a fy Z-Salgr extension (fy Z-Salgr ext) of \( A_1 \) if:

- \( (S_1) \) \( A_2 \) is a fy ext of \( A_1 \) \( (\mu_{A_1}(u) \leq \mu_{A_2}(u) \ \forall \ u \in J) \).
- \( (S_1) \) If \( A_1 \) is a fy Z-Salgr of \( J \), then \( A_2 \) is a fy Z-Salgr of \( J \).

It follows from the definition of fy \( \alpha \)-tlt, \( \mu_{A^T_{\alpha}}(u) \geq \mu_A(u) \ \forall u \in J \). This proves the following propositions.

**Proposition 3.1.** Let \( A \) be a fy Z-Salgr of a Z-algr \( J \) and \( \alpha \in [0, T] \). Then the fy \( \alpha \)-tlt \( A^T_{\alpha} \) of \( A \) is a fy Z-Salgr ext of \( A \).

**Proposition 3.2.** Arbitrary intersection of fy Z-Salgr exts of a fy set \( A \) of a Z-algr \( J \) is a fy Z-Salgr ext of \( A \).

**Definition 3.3.** For a fy set \( A \) of a Z-algr \( J \), \( \alpha \in [0, T] \) and \( t \in [0, 1] \) with \( t \geq \alpha \), we define the upper level subset of \( A^T_{\alpha} \) as \( U_{\alpha}(\mu_A; t) = \{ u \in J | \mu_A(u) \geq t - \alpha \} \).

**Proposition 3.3.** Let \( A \) be a fy set of a Z-algr \( J \) and \( \alpha \in [0, T] \). Then the fy \( \alpha \)-tlt \( A^T_{\alpha} \) of \( A \) is a fy Z-Salgr of \( J \) \( \iff \) \( U_{\alpha}(\mu_A; t) \) is a Z-Salgr of \( J \), \( \forall t \in Im(A) \) with \( t \geq \alpha \).

**Proposition 3.4.** Let \( A \) be a fy Z-Salgr of a Z-algr \( J \) and \( \alpha, \lambda \in [0, T] \). If \( \alpha \geq \lambda \), then the fy \( \alpha \)-tlt \( A^T_{\alpha} \) of \( A \) is a fy Z-Salgr ext of the fy \( \lambda \)-tlt \( A^T_{\lambda} \) of \( A \).
Proposition 3.5. Let $A$ be a fy Z-Salgr of a Z-algr $J$ and $\lambda \in [0, T]$. For every fy Z-Salgr ext $B$ of the fy $\lambda$-tlt $A^\lambda_T$ of $A$, $\exists \alpha \in [0, T] \ni \alpha \geq \lambda$ and $B$ is a fy Z-Salgr ext of the fy $\alpha$-tlt $A^\alpha_T$ of $A$.

Definition 3.4. Let $A$ be a fy set of a Z-algr $J$ and $\beta \in (0, 1]$. A fy $\beta$-multiplication (fy $\beta$-mlc) $A^M_\beta$ of $A$ with msfn $\mu_{A^M_\beta} : J \to [0, 1]$ is defined by $\mu_{A^M_\beta}(u) = \beta \cdot \mu_A(u) \forall u \in J$.

Example 2. Consider a Z-algr $J=\{0, s, p, g\}$ and a fy Z-Salgr $A$ of $J$ as in Example 3.2. Then $A^M_0$ is a fy Z-Salgr of $A$.

Proposition 3.6. If $A$ is a fy Z-Salgr of a Z-algr $J$, then the fy $\beta$-mlc $A^M_\beta$ of $A$ is a fy Z-Salgr of $J$ for all $\beta \in (0, 1]$.

Proposition 3.7. For any fy set $A$ of Z-algr $J$, $A$ is a fy Z-Salgr of $J$ iff $A^M_\beta$ is a fy Z-Salgr of $J$, $\forall \beta \in (0, 1]$.

Proposition 3.8. Let $A$ be a fy set of a Z-algr $J$, $\alpha \in [0, T]$ and $\beta \in (0, 1]$. Then every fy $\alpha$-tlt $A^\alpha_T$ of $A$ is a fy Z-Salgr ext of the fy $\beta$-mlc $A^M_\beta$ of $A$.

Proof. $A^\alpha_T$ is a fy ext of $A^M_\beta$, since

$$\mu_{A^\alpha_T}(u) = \mu_A(u) + \alpha \geq \mu_A(u) \geq \beta \cdot \mu_A(u) = \mu_{A^M_\beta}(u) \forall u \in J.$$ 

If $A^M_\beta$ is a fy Z-Salgr of $J$. Then $A$ is a fy Z-Salgr of $J$ by Proposition 3.14. It follows from Theorem 3.3 that $A^\alpha_T$ is a fy Z-Salgr of $J$ $\forall \alpha \in [0, T]$. $\square$

Theorem 3.2. Let $A$ be a fy set of Z-algr $J$ and $\alpha \in [0, T]$. Then the fy $\alpha$-tlt $A^\alpha_T$ of $A$ is a fy Z-idl of $J$ $\iff$ $A$ is a fy Z-idl of $J$.

Definition 3.5. When $A_1$, $A_2$ are fy sets of Z-algr $J$, $A_2$ is called a fy Z-idl ext of $A_1$ if:

$(I_1)$ $A_2$ is a fy ext of $A_1$.

$(I_2)$ If $A_1$ is a fy Z-idl of $J$, then $A_2$ is a fy Z-idl of $J$.

Proposition 3.9. Let $A$ be a fy Z-idl of a Z-algr $J$ and $\alpha, \gamma \in [0, T]$. If $\alpha \geq \gamma$, then the fy $\alpha$-tlt $A^\alpha_T$ of $A$ is a fy Z-idl ext of the fy $\gamma$-tlt $A^\gamma_T$ of $A$.

Proposition 3.10. Let $A$ be a fy Z-idl of a Z-algr $J$ and $\gamma \in [0, T]$. For every fy Z-idl ext $B$ of the fy $\gamma$-tlt $A^\gamma_T$ of $A$, $\exists \alpha \in [0, T] \ni \alpha \geq \gamma$ and $B$ is a fy Z-idl ext of the fy $\alpha$-tlt $A^\alpha_T$ of $A$. 
**Proposition 3.11.** Let $A$ be a fuzzy Z-idl of a Z-algr $J$ and $\alpha \in [0, T]$. Then the fuzzy $\alpha$-tilt $A^T_\alpha$ of $A$ is a fuzzy Z-idl ext of $A$.

**Proposition 3.12.** Arbitrary intersection of fuzzy Z-idl ext of a fuzzy Z-idl $A$ of a Z-algr $J$ is also a fuzzy Z-idl ext of $A$.

**Theorem 3.3.** For $\alpha \in [0, T]$, let $A^T_\alpha$ be the fuzzy $\alpha$-tilt of a fuzzy set $A$ of a Z-algr $J$. Then $A^T_\alpha$ is a fuzzy Z-idl of $J$ $\iff \forall t \in \text{Im}(A), t > \alpha \Rightarrow U_\alpha(\mu_A; t)$ is a fuzzy Z-idl of $J$.

**Proposition 3.13.** For any fuzzy set $A$ of a Z-algr $J$, $A$ is a fuzzy Z-idl of $J$ $\iff \forall \beta \in (0, 1]$, the fuzzy $\beta$-mhc $A^M_\beta$ of $A$ is a fuzzy Z-idl of $J$.

**Proposition 3.14.** Let $A$ be a fuzzy set of a Z-algr $J$, $\alpha \in [0, T]$ and $\beta \in (0, 1]$. Then every fuzzy $\alpha$-tlc $A^T_\alpha$ of $A$ is a fuzzy Z-idl ext of the fuzzy $\beta$-mhc $A^M_\beta$ of $A$.

### 4. Conclusion

In this article, we have introduced fuzzy $\alpha$-tilts and fuzzy $\beta$-multiplcations of Z-algrs and discussed their properties. We extend this concept in our research work.

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