ON EXTENSION OF PRIME RADICAL IN 2-PRIMAL NEAR-RINGS

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Abstract. In this paper, we study some characterizations of prime radical in 2-primal near-ring and introduce the notions of $\mathcal{P}_N$-Baer ideals and strongly $\mathcal{P}_N$-Baer ideals in 2-primal near-ring. Some equivalent conditions are established for $\mathcal{P}_N$-Baer ideal to be a strongly $\mathcal{P}_N$-Baer ideal.

1. Preliminaries

Throughout this paper, $\mathcal{N}$ denotes a zero symmetric right near-ring and all prime ideals are assumed to be proper in $\mathcal{N}$. For any undefined concepts and notations, we refer to Pilz [6]. Let $\mathcal{P}_\mathcal{N}$ denote the prime radical of $\mathcal{N}$, for any ideal $L$ of $\mathcal{N}$, $P(L)$ denote the prime radical of $L$ and $\mathcal{N}(\mathcal{N})$ the set of nilpotent elements of $\mathcal{N}$. An ideal $P$, of $\mathcal{N}$ is prime if for any ideals $U, V$ of $\mathcal{N}$, $UV \subseteq P$, implies $U \subseteq P$ or $V \subseteq P$. An ideal $M$ of $\mathcal{N}$ is semiprime ideal if for an ideal $K$ of $\mathcal{N}$, $K^2 \subseteq M$ implies $K \subseteq M$. An ideal $J$ of $\mathcal{N}$ is completely prime if for any $u', v' \in \mathcal{N}$, $u'v' \in J$ implies either $u' \in J$ or $v' \in J$. An ideal $J$ of $\mathcal{N}$ is completely semiprime if for any $u \in \mathcal{N}$, $u^2 \in J$ implies $u \in J$. For any non-empty subsets $R, S$ of $\mathcal{N}$, we denote the set $\{n \in \mathcal{N} : nS \subseteq R\}$ as $< R : S >$. For every ideal $Q_i$ and $K \subseteq \mathcal{N}$, $< Q_i : K >$ is maximal element among $\{< Q_i : Q_1 > : Q_1 \subseteq \mathcal{N}, < Q_i : Q_1 > \neq \mathcal{N}\}$ if and only if $< Q_i : K > \neq \mathcal{N}$ and $< Q_i : K > \subseteq < Q_i : T > \neq \mathcal{N}$ implies that $< Q_i : K > = < Q_i : T >$ for any subset $T$ of $\mathcal{N}$ [3]. If $< I_1 : K_1 >$ is the maximal element among

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\{< I_1 : K_1 > : K_1 \subseteq \mathcal{N}, < I_1 : K_1 > \neq \mathcal{N}\}, then there is \(c_1 \in \mathcal{N}\) with \(c_1 \notin < I_1 : K_1 >\) which implies \(c_1 k_1 \notin I_1\) and \(< I_1 : K_1 > \subseteq < I_1 : k_1 > \neq \mathcal{N}\) for some \(k_1 \in K_1 \setminus I_1\). So \(< I_1 : K_1 > = < I_1 : k_1 >\). If \(\mathcal{N}\) has only one nilpotent element 0, then \(\mathcal{N}\) is reduced. If \(\mathcal{P}_{\mathcal{N}} = \mathcal{N}(\mathcal{N})\), then \(\mathcal{N}\) is called 2-primal, see [1]. Clearly, every reduced near-ring is 2-primal, but 2-primal near-rings are not necessarily to be reduced, see Example 1.1 of [4]. If \(\mathcal{N}\) is 2-primal, then \(\mathcal{P}_{\mathcal{N}}\) is a completely semiprime ideal.

In [7], T. P. Speed has introduced the notion of Baer ideals in a commutative baer ring and later in [5], C. Jayaram has generalized baer ideals to a commutative semiprime ring and investigated properties of baer rings, regular rings and quasi-regular rings by using baer ideals.

Following [5], an ideal \(J\) of \(\mathcal{N}\) with \(\mathcal{P}_{\mathcal{N}} \subseteq J\) is \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal if \(x \in J\) implies \(< \mathcal{P}_{\mathcal{N}} : < \mathcal{P}_{\mathcal{N}} : x >/J\). Also, an ideal \(J\) of \(\mathcal{N}\) with \(\mathcal{P}_{\mathcal{N}} \subseteq J\) is strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal if for any \(a_1, b_1, c_1 \in \mathcal{N}\), \(< \mathcal{P}_{\mathcal{N}} : a_1 > \cap < \mathcal{P}_{\mathcal{N}} : b_1 > = < \mathcal{P}_{\mathcal{N}} : c_1 >\) and \(a_1, b_1 \in J\) imply \(c_1 \in J\). A subset \(M(\neq \phi)\) of \(\mathcal{N}\) is a multiplicative subset if \(0 \notin M\) and for \(a_1, b_1 \in M\) implies \(a_1 b_1 \in M\). Let \(D = \{c_1 \in \mathcal{N} : < \mathcal{P}_{\mathcal{N}} : c_1 >/\mathcal{P}_{\mathcal{N}}\}\). Then \(\mathcal{P}_{\mathcal{N}} \cap D = \phi\) and \(D\) is a multiplicative closed subset of \(\mathcal{N}\). For any multiplicative closed subset \(S\) of \(\mathcal{N}\), we define \(O(S) = \{c_1 \in \mathcal{N} : c_1 s \in \mathcal{P}_{\mathcal{N}}\text{ for some } s \in S\}\). For each multiplicative closed subset \(S\) of \(\mathcal{N}\), \(O(S)\) is a \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal of \(\mathcal{N}\). An ideal \(J\) of \(\mathcal{N}\) is an \(\mathcal{P}_{\mathcal{N}}\)-ideal if there exists a multiplicative subset \(M_1\) of \(\mathcal{N}\) such that \(J = O(M_1)\). Let \(I\) and \(J\) be ideals of \(\mathcal{N}\) with \(J \subseteq I, I\) is a \(J\)-ideal of \(\mathcal{N}\) if \(I = O(M_1)\) for some multiplicative subset \(M_1\) of \(\mathcal{N}\). If \(\mathcal{N}\) is 2-primal, then each minimal prime ideal is \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal. Indeed, if \(c_1 \in P_1\), where \(P_1\) is minimal prime, then, by Theorem 3.5 of [4], \(< \mathcal{P}_{\mathcal{N}} : c_1 >/\mathcal{P}_{\mathcal{N}}\) which implies \(< \mathcal{P}_{\mathcal{N}} : c_1 >/\mathcal{P}_{\mathcal{N}}\subseteq P_1\), so \(P_1\) is \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal. Every \(\mathcal{P}_{\mathcal{N}}\)-ideal is a strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal, and every strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal of \(\mathcal{N}\) is a \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal, see Lemma 2.2. For any subset \(T\) of \(\mathcal{N}\), \(< \mathcal{P}_{\mathcal{N}} : T >\) is a strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal of \(\mathcal{N}\).

Clearly intersection of strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideals (resp., \(\mathcal{P}_{\mathcal{N}}\)-Baer ideals) is again a strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal (resp., \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal). It should be noted that our definition of strongly \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal( \(\mathcal{P}_{\mathcal{N}}\)-Baer ideal) will coincide with that of Jayaram (1984) in a commutative semiprime ring.
2. Main Results

Theorem 2.1. If $\mathcal{N}$ is zero-symmetric and if $P''$ is an ideal of $\mathcal{N}$, then the statements given below are equivalent:

(i) $P''$ is prime,
(ii) For any $c_1, c_2 \in \mathcal{N}$, $c_1 < c_2 \subseteq P''$ implies $c_1 \in P''$ or $c_2 \in P''$.
(iii) If $A_1, A_2, ..., A_n$ are ideals of $\mathcal{N}$, then $A_1 A_2 ... A_n \subseteq P''$ implies $A_i \subseteq P''$ for some $i$.

Proof.

(i) $\Rightarrow$ (ii) Suppose $c_1 < c_2 \subseteq P''$ for some $c_1, c_2 \in \mathcal{N}$. Then $c_1 \in (P'' : c_1 >)$ and $c_2 \in (P'' : c_2 >)$.

Since $(P'' : c_1 >)$ is an ideal, $(P'' : c_2 >)$ is an ideal, and hence $c_1 < c_2 > \subseteq P''$ which implies $c_1 \subseteq P''$ or $c_2 \subseteq P''$.

(ii) $\Rightarrow$ (i) Let $A_1, A_2, ..., A_n$ be ideals of $\mathcal{N}$ with $A_1 A_2 ... A_n \subseteq P''$ and suppose that $A_n \notin P''$. We claim that $A_1 A_2 ... A_{n-1} \subseteq P''$. Let $c_1 \in A_1 A_2 ... A_{n-1}$ and let $c_2 \in A_n \setminus P''$. Then $c_1 < c_2 > \subseteq P''$. Since $c_2 \notin P''$ by (ii), we have $c_1 \in P''$. Thus $A_1 A_2 ... A_{n-1} \subseteq P''$. Suppose $A_{n-1} \notin P''$. Then as earlier we can show that $A_1 A_2 ... A_{n-2} \subseteq P''$. Proceeding in this way we get (iii).

(iii) $\Rightarrow$ (i) It is obvious. \hfill $\Box$

Lemma 2.1. If $\mathcal{N}$ is 2-primal, then for any $Z' \subseteq \mathcal{N}$; $c_1, c_2 \in \mathcal{N}$, we have

(i) $\langle \mathcal{P}_N : Z' \rangle = \langle \mathcal{P}_N : Z' >\rangle$ and $Z' \subseteq \langle \mathcal{P}_N : \langle \mathcal{P}_N : Z' \rangle \rangle$,

(ii) $\langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 c_2 \rangle = \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_2 \rangle \rangle$,

(iii) If $c_1 c_2 \in \mathcal{P}_N$, then $\langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_2 \rangle \rangle$.

Proof.

(i) For $t \in \mathcal{N}$, $t \in \langle \mathcal{P}_N : Z' \rangle \iff a Z' \subseteq \mathcal{P}_N \iff a < Z' \subseteq \mathcal{P}_N \iff a \in \langle \mathcal{P}_N : Z' \rangle$. Also if $c_1 \in Z'$, then for any $t \in \langle \mathcal{P}_N : Z' \rangle$, we have $tc_1 \in \mathcal{P}_N$ which implies $c_1 \in \langle \mathcal{P}_N : \langle \mathcal{P}_N : Z' \rangle \rangle$.

(ii) Clearly $\langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_2 \rangle = \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 c_2 \rangle \rangle$. If $t \in \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 c_2 \rangle$, then $t \in \langle \mathcal{P}_N : c_1 c_2 \rangle \subseteq \mathcal{P}_N$, so for $a \in \langle \mathcal{P}_N : c_1 \rangle \subseteq \mathcal{P}_N : c_1 c_2$ and $b \in \mathcal{P}_N : c_2 \subseteq \mathcal{P}_N : c_1 c_2$, $ta \in \mathcal{P}_N$ and $tb \in \mathcal{P}_N$ imply $t \in \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_2 \rangle \rangle$. So $\langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 c_2 \rangle = \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : \langle \mathcal{P}_N : c_2 \rangle \rangle$.

(iii) If $t \in \langle \mathcal{P}_N : c_1 \rangle \cap \langle \mathcal{P}_N : c_2 \rangle$, then $tc_1 \in \mathcal{P}_N$ and $tc_2 \in \mathcal{P}_N$ which imply $t(c_1 + c_2) \in \mathcal{P}_N$. Thus $t \in \langle \mathcal{P}_N : c_1 + c_2 \rangle$. If $s \in \langle \mathcal{P}_N : c_1 + c_2 \rangle$, then
Theorem 2.4. If \( N \) is completely semiprime, we have \( \mathcal{P}_N \) is 2-primal, then the statements given below are equivalent:

(i) \( \mathcal{P}_N : S > \) is maximal among \( T \),

(ii) \( \mathcal{P}_N : S > \) is completely prime,

(iii) \( \mathcal{P}_N : S > \) is minimal prime.

Proof.

(i) \( \Rightarrow \) (ii) By assumption, \( \exists y \in S \setminus \mathcal{P}_N : < \mathcal{P}_N : S > = < \mathcal{P}_N : y >= < \mathcal{P}_N : y^2 > \). Since \( < \mathcal{P}_N : y > \subseteq < \mathcal{P}_N : y^2 > \) and \( y^3 \notin \mathcal{P}_N \), \( < \mathcal{P}_N : y > = < \mathcal{P}_N : y^2 > \). If \( ab \in < \mathcal{P}_N : y > \) and \( a \notin < \mathcal{P}_N : y > \) for some \( a, b \in N \), then \( < \mathcal{P}_N : ya > \neq \mathcal{N} \). By the maximality of \( < \mathcal{P}_N : y > \), we have \( b \in < \mathcal{P}_N : ya > = < \mathcal{P}_N : y > \).

(ii) \( \Rightarrow \) (iii) Suppose that \( Q \) is prime with \( Q \subseteq < \mathcal{P}_N : S > \). Let \( y \in S \setminus \mathcal{P}_N \) and \( a \in < \mathcal{P}_N : S > \). Then \( a > y \in \mathcal{P}_N \subseteq Q \). Since \( y^2 \notin \mathcal{P}_N \), we have \( a \notin Q \). So \( Q = < \mathcal{P}_N : S > \). (iii) \( \Rightarrow \) (ii) It follows from Corollary 1.3 of [2].

Theorem 2.3. If \( N \) is 2-primal, then the maximality and minimality conditions of the elements on the set \( T = \{ < \mathcal{P}_N : A_1 > : A_1 \subseteq N, < \mathcal{P}_N : A_1 > \neq \mathcal{N} \} \) are coincide.

Proof. Suppose a.c.c. holds on the set \( T \) and let \( < \mathcal{P}_N : X_1 > \supseteq < \mathcal{P}_N : X_2 > \supseteq \ldots \) be a descending chain on the set \( T \). Then \( < \mathcal{P}_N : < \mathcal{P}_N : X_1 > \supseteq < \mathcal{P}_N : < \mathcal{P}_N : X_2 > \supseteq \ldots \) is an ascending chain, which ends after finite steps.

Since \( < \mathcal{P}_N : < \mathcal{P}_N : X_1 > \supseteq < \mathcal{P}_N : X_1 > \), so the descending chain \( < \mathcal{P}_N : < \mathcal{P}_N : X_1 > \supseteq < \mathcal{P}_N : X_2 > \supseteq \ldots \) or the chain ends after finite steps. Similarly we can prove the converse part.

Theorem 2.4. For any \( N \neq \mathcal{P}_N \), the statements given below are equivalent:
(i) \( \mathcal{N} \) is 2-primal and, for every \( c_1 \in \mathcal{N} \setminus \mathcal{P}_\mathcal{N}, \ < \mathcal{P}_\mathcal{N}: c_1 > \) is contained in some maximal element among \( T = \{ < \mathcal{P}_\mathcal{N}: X_n > : X_n \subseteq \mathcal{N}, \ < \mathcal{P}_\mathcal{N}: X_n > \neq \mathcal{N} \} \),

(ii) The number of minimal completely prime ideals \( P_i, \ i = 1, 2, \ldots, n \) with \( \bigcap_{i=1}^{n} P_i = \mathcal{P}_\mathcal{N} \) of \( \mathcal{N} \) is finite.

Proof.

(i) \( \Rightarrow \) (ii) Assume that \( c_1c_2 \in \mathcal{P}_\mathcal{N} \) for some \( c_1 \in \mathcal{N} \setminus \mathcal{P}_\mathcal{N} \) and \( c_2 \in \mathcal{N} \setminus \mathcal{P}_\mathcal{N} \). Then there is a maximal element \( < \mathcal{P}_\mathcal{N}: c_3 > \) in \( T \) such that \( c_3 \in \mathcal{N} \setminus \mathcal{P}_\mathcal{N} \) and \( c_1 \in < \mathcal{P}_\mathcal{N}: c_2 > \subseteq < \mathcal{P}_\mathcal{N}: c_3 > \). By Theorem 2.2, \( < \mathcal{P}_\mathcal{N}: c_3 > \) is completely prime ideal. Consider the set of all distinct minimal completely prime ideals \( P_\alpha \) of \( \mathcal{N} \) where \( P_\alpha = < \mathcal{P}_\mathcal{N}: z_\alpha > (\alpha \in I) \) and \( z_\alpha \in \mathcal{N} \setminus \mathcal{P}_\mathcal{N} \). Let \( P = \bigcap_{\alpha \in I} P_\alpha \). Then \( z_\alpha \in < \mathcal{P}_\mathcal{N}: P_\alpha > \) and \( < \mathcal{P}_\mathcal{N}: P_\alpha > \subseteq < \mathcal{P}_\mathcal{N}: P > \) for all \( \alpha \in I \).

We now claim that \( \mathcal{P}_\mathcal{N} = P \). If not, then there is a maximal element \( < \mathcal{P}_\mathcal{N}: z_\beta > \) in \( T \) with \( z_\beta \notin \mathcal{N} \setminus \mathcal{P}_\mathcal{N} \) and \( < \mathcal{P}_\mathcal{N}: P > \subseteq < \mathcal{P}_\mathcal{N}: z_\beta > \). So \( P_\beta = < \mathcal{P}_\mathcal{N}: z_\beta > \) for some \( \beta \in I \), but \( z_\beta \notin < \mathcal{P}_\mathcal{N}: P > \), we have \( z_\beta^2 \in \mathcal{P}_\mathcal{N} < \mathcal{P}_\mathcal{N}: P > \subseteq z_\beta < \mathcal{P}_\mathcal{N}: z_\beta > \), a contradiction. So \( P = \bigcap_{\alpha \in I} P_\alpha = \mathcal{P}_\mathcal{N} \). We now prove that \( |I| \) is finite. If not, then for some \( \alpha_1 \in I \),

\(< \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \) is not contained in all \( < \mathcal{P}_\mathcal{N}: z_\alpha > \) which implies \( z_{\alpha_1}z_\alpha \) for all \( \alpha (\neq 1) \in I \). Take some \( \alpha_2 \in I \), \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \subseteq < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \) which implies \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \subseteq < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \) \( \cap < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \). If \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \cap < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \neq \mathcal{P}_\mathcal{N} \), then we have a descending chain \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \supseteq < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \cap < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \) \( \cap < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \ldots \)\. If \( \ell(z_{\alpha_1} + z_{\alpha_2}) \in \mathcal{P}_\mathcal{N} \), then \( \ell z_{\alpha_1} \in \mathcal{P}_\mathcal{N} \) and \( \ell z_{\alpha_2} \in \mathcal{P}_\mathcal{N} \). This shows that \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \cap < \mathcal{P}_\mathcal{N}: z_{\alpha_2} > \supseteq < \mathcal{P}_\mathcal{N}: z_{\alpha_1} + z_{\alpha_2} > \cdot \) So the obtained descending chain \( < \mathcal{P}_\mathcal{N}: z_{\alpha_1} > \supseteq < \mathcal{P}_\mathcal{N}: z_{\alpha_1} + z_{\alpha_2} > \supseteq \ldots \) not terminated, a contradiction to Theorem 2.3. Hence \( |I| \) is finite.

(ii) \( \Rightarrow \) (i) It is trivial as \( \mathcal{N} / \mathcal{P}_\mathcal{N} \) is reduced. \( \square \)

Lemma 2.2. Let \( \mathcal{N}_1 \) be an ideal of a 2-primal near-ring \( \mathcal{N} \) with \( \mathcal{P}_\mathcal{N} \subseteq \mathcal{N}_1 \). If \( \mathcal{N}_1 \) is strongly \( \mathcal{P}_\mathcal{N} \)-Baer ideal, then \( \mathcal{N}_1 \) is a \( \mathcal{P}_\mathcal{N} \)-Baer ideal of \( \mathcal{N} \).

Proof. Let \( \mathcal{N}_1 \) be a strongly \( \mathcal{P}_\mathcal{N} \)-Baer ideal of \( \mathcal{N} \). For \( c_1 \in \mathcal{N}_1 \) and let \( c_2 \in < \mathcal{P}_\mathcal{N}: c_1 > \). We now prove that \( < \mathcal{P}_\mathcal{N}: c_1 > = < \mathcal{P}_\mathcal{N}: c_1c_2 > \). Clearly \( < \mathcal{P}_\mathcal{N}: c_1 > \subseteq < \mathcal{P}_\mathcal{N}: c_1c_2 > \). Let \( a \in < \mathcal{P}_\mathcal{N}: c_1c_2 > \). Then \( ac_2 \in < \mathcal{P}_\mathcal{N}: c_1c_2 > \).
Corollary 2.1. For every strongly $P_N : c_1 >$. Since $c_2 < P_N : c_1 \subseteq P_N$, we get $ac_2^2 \in P_N$ implies $ac_2 \in P_N$, so $a \in P_N : c_2 >$. Thus $P_N : c_2 \Rightarrow P_N : c_1 c_2 >$ and hence $c_2 \in N_1$. □

Lemma 2.3. If $Q$ is an ideal of a 2-primal near-ring $N$ with $P_N \subseteq Q$, then the statements given below are equivalent:

(i) $Q$ is $P_N$-Baer ideal,

(ii) For any $c_1, c_2 \in N$, $P_N : c_1 \Rightarrow P_N : c_2$ and $c_1 \in Q$ imply $c_2 \in Q$,

(iii) $Q = \bigcup_{c_1 \in Q} P_N : < P_N : c_1 >$.

Proof.

(i) ⇒ (ii) and (iii) ⇒ (i) are evident.

(ii) ⇒ (iii) For any $c_1 \in Q$ and $c_2 \in P_N : < P_N : c_1 >$, we have $P_N : c_1 \Rightarrow P_N : c_2$. Thus $P_N : c_2 \Rightarrow P_N : c_1 > \cup P_N : c_1 = P_N : c_2$ as $c_2 < P_N : c_1 \subseteq P_N$. Since $c_1 c_2 \in Q$, we have $c_2 \in Q$. So, $\bigcup_{c_1 \in Q} P_N : < P_N : c_1 > \subseteq Q$. Since for any $c_1 \in N$, we have $c_1 \in P_N : < P_N : c_1 >$. Thus $Q \subseteq \bigcup_{c_1 \in Q} P_N : < P_N : c_1 >$ and hence $Q = \bigcup_{c_1 \in Q} P_N : < P_N : c_1 >$. □

Lemma 2.4. If $Q$ is a $P_N$-Baer ideal of a 2-primal near-ring $N$, then $Q = P_N(Q)$.

Proof. Let $c_1 \in P_N(Q)$. Then, by Proposition 2.94 of [6], we can find a positive integer $n$ such that $c_1^n \in Q$. Since $N$ is 2-primal, we get $P_N : c_1 = P_N : c_1^n$. By Lemma 2.3, we have $c_1 \in Q$. Thus $P_N(Q) \subseteq Q$ and hence $Q = P_N(Q)$. □

Corollary 2.1. For every strongly $P_N$-Baer ideal $Q$ of a 2-primal near-ring $N$, we have $Q = P_N(Q)$.

Proof. It follows from Lemma 2.2 and Lemma 2.4. □

Theorem 2.5. If $I_1$ is a reflexive ideal of $N$ and $P''$ is prime with $I_1 \subseteq P''$, then the statements given below are equivalent:

(i) $P''$ is a minimal prime,

(ii) For every $a \in P''$, there exist $x_i \in N \setminus P''$ such that $a t_0 x_1 a t_1 x_2 a t_2 x_3 \cdots x_n a t_n \in I_1$, where $t_i$'s are positive integers with $t_0$ and $t_n$ allowed to be zero.
Proof.

(i) $\Rightarrow$ (ii) Let $a \in P''$ and $T = \{a^{i_0}x_1a^{i_1}x_2a^{i_2}x_3...x_na^{i_n},$ where $x_i \in N\backslash P''$ and $t_i'$s are the positive integers with $t_0$ and $t_n$ allowed to be zero}. Then $F = T \cup (N\backslash P'')$ is a multiplicative closed subset of $N$. If $I_1 \cap F = \phi$, then, by Proposition 2.1.6 of [1], there exists a proper maximal ideal $M_1$ with $M_1 \cap F = \phi$. Since $a \not \in M_1$, we have $M_1 + <a> = N$ which implies $b + c = 1$ for some $b \in M_1$ and $c \leq a$. Since $a \in P''$, we have $b \in N\backslash P''$. So $b \in M_1 \cap F \neq \{\phi}\}, a$ contradiction. Thus $I_1 \cap F = \phi$ and hence $I_1 \cap T \neq \{\phi\}$.

(ii) $\Rightarrow$ (i) Suppose that $K$ is a prime ideal with $I_1 \subseteq K \subseteq P''$. Then for any $a \in P''$, there are $x_i \in N\backslash P''$ such that $a^{i_0}x_1a^{i_1}x_2a^{i_2}x_3...x_na^{i_n} \in I_1$ where $t_i'$s are positive integers with $t_0$ and $t_n$ allowed to be zero. Since $I_1$ is reflexive ideal, we have $<a> = x_1 <a> x_2 <a> ... <x_n <a> t_n \subseteq I_1 \subseteq K$ which implies $a \in K$. Thus $P'' \subseteq K$ and hence $P''$ is a minimal prime.

Lemma 2.5. Let $N$ be a 2-primal near-ring and $K$ a $P_N$-Baer ideal (resp., strongly $P_N$-Baer ideal) of $N$. Then each minimal prime ideal $P$ of $N$ containing $K$ is a $P_N$-Baer ideal (resp., strongly $P_N$-Baer ideal) of $N$.

Proof. Suppose that $K$ is a $P_N$-Baer ideal and $P$ is minimal prime containing $K$. Let $<P_N : c_1 > = <P_N : c_2 >$ and $c_1 \in P$. Then by Theorem 2.5, there exist $x_i \in N\backslash P$ such that $c_1^{i_0}x_1c_1^{i_1}x_2c_1^{i_2}x_3...x_nc_1^{i_n} \in K$ where $t_i'$s are positive integers with $t_0$ and $t_n$ allowed to be zero. Since $<P_N : x_1x_2...x_nc_1 > = <P_N : x_1c_1^{i_0}x_1c_1^{i_1}x_2c_1^{i_2}x_3...x_nc_1^{i_n} >$ and $K$ is a $P_N$-Baer ideal, we have $x_1x_2...x_nc_1 \in K$ and so $x_1x_2...x_n <c_2 \in P$. Consequently $c_2 \in P$ as $x_i'$s are not in $P$. Therefore $P$ is a $P_N$-Baer ideal.

Corollary 2.2. Let $N$ be a 2-primal near-ring. Then every $P_N$-Baer ideal (resp., strongly $P_N$-Baer ideal) of $N$ is the intersection of every prime $P_N$-Baer ideals (resp., prime strongly $P_N$-Baer ideals) containing it.

Proof. It is evident from Lemma 2.4 and Lemma 2.5.

Lemma 2.6. Let $N$ be a 2-primal near-ring and $Q$ be a $P_N$-ideal of $N$. Then every minimal prime ideal belonging to $Q$ is a minimal prime ideal of $N$.

Proof. Let $Q$ be a $P_N$-ideal and $P$ a minimal prime ideal belonging to $Q$. Then $Q = O(K)$ for some multiplicative subset $K$ of $N$ and $Q$ is reflexive. By Theorem 2.5, we claim that for each $q \in P$, there exist $x_i \in N\backslash P$ such that
Lemma 2.7. Suppose that for each \( u \in \mathcal{N} \), there is \( v \in \mathcal{N} \) such that \(< \mathcal{P}_N : u > =< \mathcal{P}_N : v > \). Then every \( \mathcal{P}_N \)-Baer ideal is strongly \( \mathcal{P}_N \)-Baer ideal.

Proof. Assume that for each \( u \in \mathcal{N} \), there is \( v \in \mathcal{N} \) with \(< \mathcal{P}_N : u > =< \mathcal{P}_N : v > \). Let \( Q \) be a \( \mathcal{P}_N \)-Baer ideal of \( \mathcal{N} \) and \(< \mathcal{P}_N : c_1 > \cap < \mathcal{P}_N : c_2 > =< \mathcal{P}_N : c_3 > \) for \( c_1, c_2 \in Q \). By assumption, there exist \( c_1', c_2' \in \mathcal{N} \) with \(< \mathcal{P}_N : c_1' > =< \mathcal{P}_N : c_1 > \) and \(< \mathcal{P}_N : c_2' > =< \mathcal{P}_N : c_2 > \). Since \( c_1 \in< \mathcal{P}_N : < \mathcal{P}_N : c_1 > \) and \( c_2 \in< \mathcal{P}_N : < \mathcal{P}_N : c_2 > \), we have \( c_1 c_1', c_2 c_2' \in \mathcal{P}_N \) and \( c_1 + c_1', c_2 + c_2' \in D \). Suppose \( c_3 \notin Q \). By Lemma 2.4, there is a prime \( \mathcal{P}_N \)-Baer ideal \( P \) of \( \mathcal{N} \) such that \( Q \subseteq P \) and \( c_3 \notin P \). Since \( c_3 c_1', c_2' \in \mathcal{P}_N \), we get \( c_1' \in P \) or \( c_2' \in P \). But in either case we have \( P \cap D \neq \phi \), as \( P \) is a \( \mathcal{P}_N \)-Baer ideal. Thus \( c_3 \in Q \) and hence \( Q \) is a strongly \( \mathcal{P}_N \)-Baer ideal. □

Theorem 2.6. If \( \mathcal{N} \) is a 2-primal with identity, then the statements given below are equivalent:

(i) Every ideal of \( \mathcal{N} \) containing \( \mathcal{P}_N \) is a \( \mathcal{P}_N \)-ideal,

(ii) Every ideal of \( \mathcal{N} \) containing \( \mathcal{P}_N \) is strongly \( \mathcal{P}_N \)-Baer ideal,

(iii) Every ideal of \( \mathcal{N} \) containing \( \mathcal{P}_N \) is \( \mathcal{P}_N \)-Baer ideal,

(iv) For any \( s, t \in \mathcal{N} \), \(< \mathcal{P}_N : s > =< \mathcal{P}_N : t > \implies s =< t > \),

(v) For any \( s \in \mathcal{N} \), we have \( s + s^2 \in \mathcal{P}_N \).

Proof.

(i) \( \Rightarrow \) (ii) Let \( K' \) be an ideal of \( \mathcal{N} \). Then \( K' = O(R') \) for some multiplicative subset \( R' \) of \( \mathcal{N} \). Let \( c_1, c_2 \in K' \) with \(< \mathcal{P}_N : c_1 > \cap < \mathcal{P}_N : c_2 > =< \mathcal{P}_N : z > \) for some \( z \in \mathcal{N} \). Then \( c_1 s_1, c_2 s_2 \in \mathcal{P}_N \) for some \( s_1, s_2 \in R' \). Since \( s_1, s_2 \in R' \) and \( s_1 s_2 \in< \mathcal{P}_N : c_1 > \cap < \mathcal{P}_N : c_2 > \), we have \( (s_1 s_2) z \in \mathcal{P}_N \). Thus \( z = O(R') = K' \) and hence \( K' \) is a strongly \( \mathcal{P}_N \)-Baer ideal of \( \mathcal{N} \).

(ii) \( \Rightarrow \) (iii) It is evident from the fact that each \( \mathcal{P}_N \)-ideal of \( \mathcal{N} \) is a strongly \( \mathcal{P}_N \)-Baer ideal of \( \mathcal{N} \).
(iii) $\Rightarrow$ (iv) It is trivial as every strongly $\mathcal{P}_N$-Baer ideal of $\mathcal{N}$ is $\mathcal{P}_N$-Baer ideal of $\mathcal{N}$.

(iv) $\Rightarrow$ (v) For each $s \in \mathcal{N}$, $< \mathcal{P}_N : s > = < \mathcal{P}_N : s^2 >$. By (iv), $< s > = < s^2 >$ which implies $s + s^2 < s > + < s^2 > \subseteq \mathcal{P}_N$.

(v) $\Rightarrow$ (i) Let $I_1$ be an ideal of $\mathcal{N}$ with $\mathcal{P}_N \subseteq I_1$ and let $t \in I_1$. Then $(1 + t)t \in \mathcal{P}_N$. Take $I_s = \{ x \in \mathcal{N} : < \mathcal{P}_N : z > \subseteq < \mathcal{P}_N : x > \text{ for some } z \in I_1 \}$. Then $I_s$ is a multiplicative closed subset of $\mathcal{N}$ and $< \mathcal{P}_N : t > \subseteq < \mathcal{P}_N : \mathcal{P}_N : 1 + t >$ which imply $1 + t \in I_s$ and $t \in O(I_s)$, so $I_1 \subseteq O(I_s)$. Let $r \in O(I_s)$. Then $rs \in \mathcal{P}_N$ for some $s \in I_s$, with $< \mathcal{P}_N : z > \subseteq < \mathcal{P}_N : \mathcal{P}_N : s >$ for some $z \in I_1$. By (v), $(1 + z)z \in \mathcal{P}_N$ implies $1 + z \in < \mathcal{P}_N : < \mathcal{P}_N : s > >$. Since $r \in < \mathcal{P}_N : s >$, we have $(1 + z)r = r + zr \in \mathcal{P}_N \subseteq I_1$ which implies $r \in I_1$.

Thus $O(I_s) \subseteq I_1$ and hence $I_1$ is $\mathcal{P}_N$-ideal.  

\section*{Theorem 2.7.} If $\mathcal{N}$ is 2-primal, then the statements given below are equivalent:

(i) For any $c_1 \in \mathcal{N}$, there is $c_2 \in \mathcal{N}$ such that $< \mathcal{P}_N : < \mathcal{P}_N : c_1 >= < \mathcal{P}_N : c_2 >$,

(ii) Every $\mathcal{P}_N$-Baer ideal of $\mathcal{N}$ containing $\mathcal{P}_N$ is an $\mathcal{P}_N$-ideal,

(iii) Every strongly $\mathcal{P}_N$-Baer ideal of $\mathcal{N}$ containing $\mathcal{P}_N$ is an $\mathcal{P}_N$-ideal,

(iv) For $X \subseteq \mathcal{N}$, $< \mathcal{P}_N : X >$ is an $\mathcal{P}_N$-ideal.

\section*{Proof.}

(i) $\Rightarrow$ (ii) It is evident from Lemma 2.7 and Theorem 2.6. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. (iv) $\Rightarrow$ (i) Let $n \in I$. Then by (iv), $< \mathcal{P}_N : < \mathcal{P}_N : n > >= O(S)$ for some multiplicative subset $S$ of $\mathcal{N}$ and $ns \in \mathcal{P}_N$ for some $s \in S$ which imply $< \mathcal{P}_N : < \mathcal{P}_N : n > \subseteq < \mathcal{P}_N : y >$. Also $< \mathcal{P}_N : y > \subseteq O(S) = < \mathcal{P}_N : < \mathcal{P}_N : n > >$. Therefore $< \mathcal{P}_N : < \mathcal{P}_N : n > >= < \mathcal{P}_N : y >$. 

\section*{References}


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