SURVEY ON INCORPORATING FRACTIONAL DERIVATIVES IN IMAGE DENOISING

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ABSTRACT. Gaussian and Impulse noises are built-in feature of images. It infects and reduces the nature value of the image badly. This survey article produces a detailed and latest view of denoising methods based on fractional derivatives applied to damaged images and outline the growth that has been construct over the years in all supplications requiring image processing. Fractional calculus based denoising model is registered on each kernel of the noisy image. The results of this study highlight the need to practice fractional derivatives based denoising models will be foremost relevant to their research need and the demand field where such methods are inspected for performance.

1. INTRODUCTION

1.1. Review. The phrase noise is used to report undesired one that may degrade an image. Gaussian noise and speckle noise corruptions medical images such as ultra sound images, MRI (Magnetic Resonance Image) images, X-ray images, CT(Computed Tomography) images, OCT(Optical Coherence Tomography) images, text images, rich texture images and general images etc. Gaussian noise is additive and independent also it is calculated from Gaussian distribution function and it is induced by three usual noises i.e amplifier noise, shot noise and grain noise. Accordingly, the noisy image can be shown as

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\[ o(i, j) = f(i, j) + n(i, j), \]

where \( o(i,j) \) denotes the observed noisy image, \( f(i,j) \) denotes the original image (noise free) and \( n(i,j) \) denotes additive noise on a pixel basis. Impulse noise is multiplicative noise and calculated from Gamma distribution function. Correspondingly, the noisy image can be expressed as

\[ o(i, j) = f(i, j) \ast n(i, j), \]

where \( o(i,j) \) represents the observed noisy image, \( f(i,j) \) represents the original image (noise free) and \( n(i,j) \) represents multiplicative noise on a pixel basis.

There are two fundamental methods proceed towards image denoising, spatial filtering methods and transform domain filtering methods. The classical approach to separate noise from image is to engage spatial filters. Spatial filters can be further classified into nonlinear and linear filters. The disadvantage of the linear model is that it is not able to protect edges in a good way; on the other hand the nonlinear models can handle edges in a much better way than linear models. The subject of fractional calculus and it is applications have acquired substantial popularity during the past decades in the fields of science and engineering like dynamics system and image processing. The sequence of image process contains three steps. First, the effective operator model, second, generalized to fractional order using Grünwald-Letnikov (GL), Riemann Liouville (RL), Caputo etc. and ultimately the numerical approximation to the fractional order operator based model will be discretized.

**Figure 1.** Schematic picture for image denoising with fractional derivatives

2. Fractional Derivatives

The goal of fractional calculus was initiated over 300 years ago. Abel, in 1823, scrutinized the generalized tauto-chrone problem and for the first time, applied
fractional calculus techniques in a physical problem. Liouville subsequently applied fractional calculus to problems in potential theory. The name fractional is usually increased to several classical procedures and operators. A little while back, the fractional order derivative has been satisfied much more awareness in engineering. The fractional order derivative can be noticed as the generalization of the integer order derivative. It has been learned by Euler, Hardy, Littlewood and Liouville etc. \cite{17} for plenty of years. Research on fractional calculus that bring in different operators, such as Riemann-Liouville, Gr\"unwald-Letnikov operators, Caputo, Riesz and Erd\‘elyi-kober evolved in the past 40 years and have continued in other areas.

2.1. Riemann-Liouville (RL) fractional derivative. Riemann and Liouville first considered the possibility of fractional calculus in 1832-1837. In this subsection we give the definition of the RL operator in the real line. The Riemann-Liouville fractional integral $I_{a+}^{\alpha} f(x)$ and $I_{b-}^{\alpha} f(x)$ of order $\alpha \in \mathbb{R}^+$ are left sided and the right sided RL integrals defined as

$$RLI_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt (x > a; a > 0)$$

and

$$RLI_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x \frac{f(t)}{(t-x)^{1-\alpha}} dt (x < b; a > 0).$$

The RL fractional derivative $D_{a+}^{\alpha} f(x)$ and $D_{b-}^{\alpha} f(x)$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$RLD_{a+}^{\alpha} f(x) = \left( \frac{d}{dt} \right)^n I_{a+}^{\alpha-n} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^x \frac{f(t)}{(x-t)^{n-\alpha+1}} dt \\
(n = [\alpha] + 1; x > a; \alpha > 0)$$

and

$$RLD_{b-}^{\alpha} f(x) = \left( \frac{-d}{dt} \right)^n I_{b-}^{\alpha-n} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{-d}{dt} \right)^n \int_x^b \frac{f(t)}{(t-x)^{n-\alpha+1}} dt \\
(n = [\alpha] + 1; x < b; \alpha > 0).$$

This RL operator has extensive properties, such as linearity, additively and commutatively, neutral element, backward compatibility and derivative of a product.

Y. Zhang et al. \cite{23} analyzed fractional differential masks placed on the RL definition not only able to keep up the low frequency smoothing area; but also upgrade the high frequency part, texture and edges and it acts as the numerical properties of fractional differentiation. It was further observed that the authors
in [9] have constructed image denoising algorithm on the orientations of 135 degrees, 90 degrees, 45 degrees, 0 degrees, 180 degrees, 315 degrees and 225 degrees. Moreover fractional integral equation which is noted as a volterra equation when used as a pixel by pixel technique is propounded in [6] as well as in this article the RL integral is defined as convolution integral on the other hand this model proposed suitable to a closed interval both from analytical and numerical way of thinking.

Continuously in [15], authors have discussed fractional Green formula, fractional Gauss formula and fractional stokes formula based on RL operator. Finally they concluded that it can be preserved the small frequency contour feature in the smooth area to the further most degree and nonlinearly retain high amount of edge detail and texture information in those areas where grey scale changes frequently and as well as nonlinearly enhance texture details in those areas where grey scale region does not clear. In [24], authors have proposed a spatial fractional telegraph equation for image denoising and interpolates between diffusion equation and wave equation. The equation will be discretized by the RL operator. Then the model can be preserved edges in the extremely oscillatory regions. The results of this model specify advantage of the provided model over the existing methods. Also in [19], authors have provided a new way of construction of denoising mask based on the RL fractional derivative which can catch both local discontinuities in intensity and locating Dirac edges. In [20] authors have presented anisotropic diffusion model based on fractional order differential RL for denoising model.

Recently in [16], authors have derived mathematical models based on numerical implementation of RL and studied the ability of protecting edges and texture informations. It is superior to that of traditional integer order computation. Also the authors have discussed how to concern fractional calculus to signal and image processing. This model can be protected edges and the characteristics from being over smoothed. There were many denoising models based on RL operator are existing in the area of image processing and signal processing.

2.2. Grunwald-Letnikov (GL) fractional derivative: Grunwald and Letnikov first studied this fractional derivative theory in 1867-1868. The GL definition can be handed-down in the growth of numerical methods. The classical GL
fractional derivative is generalized in the following expression

\[ GLD^\alpha f(t) = e^{-i\theta \alpha} \lim_{|h| \to 0} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - kh)/|h|^\alpha, \]

where \( h = |h|e^{i\theta} \) is a complex number with \( \theta \in (-\pi, \pi] \) and \( \theta \neq \pm \frac{\pi}{2} \) (the pure imaginary case will not be taken here). Expanding the binomial coefficient yields \( \binom{\alpha}{k} = \frac{\alpha!}{m!(n-m)!} \). We will consider that \( \alpha \) will be any real although the results stand valid for complex values. To keep up and give an explanation to the above formula, assume that \( \theta \) is a time and \( \theta \) is real, \( \theta = 0 \) or \( \theta = \pi \). If \( \theta = 0 \), only the present and past values are being used, while, if \( \theta = \pi \) only the present and future values are used. In general, if \( \theta = 0 \), we call equation (2.1) the forward Grünwald-Letnikov derivative,

\[ GLD^\alpha f(t) = e^{-i\theta \alpha} \lim_{|h| \to 0} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - kh)/|h|^\alpha, \]

where \( \binom{\alpha}{k} \) are the binomial coefficients. If \( \theta = \pi \) we put \( h = -|h| \) to acquire the backward Grünwald-Letnikov derivative

\[ GLD^\alpha f(t) = e^{-i\theta \pi} \lim_{|h| \to 0} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - kh)/|h|^\alpha. \]

This derivative, is accurate from a mathematical ideas and is not reasonable from a system approach since it does not indicate a backward system. The sequence in (2.2) and (2.3) converges completely and uniformly if \( x(t) \) is bounded.

The definition is clearly influenced by the nonlocal feature of fractional derivatives. The arbitrary order derivative of a function at time \( t' \) based on all values of that function in \((-\infty, t]\) and \([t, \infty)\) because of the infinite sum, backward and forward difference nature of the left and right derivatives, respectively. For numerical simulations, we find the Grundwald-Letnikov is to be very convenient.

In addition, the spatial step in the numerical application of the fractional differential operator formed on the meaning of Grunwald-Letnikov usually advanced by one. This GL operator has extensive properties, such as linearity, additively and commutatively, neutral element, backward compatibility and derivative of a product.

C. F. Hu et al. [10] approached to analyze texture characteristics, interpretation of the local texture design and issuing in the support region with the
carrying of fractional differential operator mask. This mask can ensures that enough to cover the spatial extent of local texture patterns. Y. F. Pu et al. [14] published the highlights of GL derivative in image processing. In [7] developed the applications of fractional derivative based on GL and implemented in edge detection of color image.

A new adaptive method for image deblurring or denoising is introduced by D. Chen et al. [4]. This recommended algorithm provides finer value of signal to noise ratio and visual effect than using non adaptive fractional order image restorations. It was further observe that D. Chen et al. [3] studied the fractional order total variation (TV) and proximity operator based denoising model has been produced the resulting image keep away from the blocky effect. This proximity algorithm is processed by GL operator and used for improving performance of the typical error. TV algorithm has better effect on the high level noise comparing with integer order TV algorithm. In [8] He et al. investigated GL based improved denoising model for preserving detailed features while effectively denoising the image. Recently A. Abirami, P. Prakash and K. Thangavel, [1] presented a fractional diffusion based image denoising model using CN-GL scheme for all types of images without loss of any information of textures. In [12], authors have discussed a modified GL to enhance more and detect better edge information of an image. It can be performed very flexible and comfortable. The author in [2] provided a new framework for multiplicative denoising model which is derived the underlying fractional Euler-Lagrange equations and numerical implementation combining implicit and conjugate gradients for solving the nonlinear systems. The uniqueness of the solution by this model is proved using mathematical induction method.

2.3. Caputo fractional derivative. It was studied by the Italian mathematician M. Caputo in 1967. In this subsection an alternative operator to the Riemann-Liouville operator is considered as Caputo. When solving differential equations using Caputo’s definition, it is not necessary to define the fractional order initial conditions.

The Caputo fractional derivatives of function $f(t)$ with order $\alpha$ are $CD^\alpha_{a+} f(t)$ and $CD^\alpha_{b-} f(t)$ exist almost everywhere on $[a, b]$. Then $CD^\alpha_{a+} f(t)$ and $CD^\alpha_{b-} f(t)$
represented by

\[ CD^\alpha_{a+} f(t) = RLI_{a+}^{n-\alpha} \left( \frac{d^n}{dt^n} \right) f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(t)}{(x-t)^{\alpha-n+1}} dt \]

\( (n = [\alpha] + 1; x > a; \alpha > 0) \) and

\[ CD^\alpha_{b-} f(t) = RLI_{b-}^{n-\alpha} \left( \frac{d^n}{dt^n} \right) f(t) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^n(t)}{(t-x)^{\alpha-n+1}} dt, \]

\( (n = [\alpha] + 1; x < b; \alpha > 0) \).

Caputo fractional operator is applicable for finding the solutions of initial value problem. Also the solution by Caputo operator has very better physical interpretation compare to RL operator. Caputo fractional derivative is more restrictive as well as it requires the existence of \( n^{th} \) derivative of the function. This Caputo operator has extensive properties, such as linearity, additively and commutatively, neutral element and derivative of a product.

In [13], authors introduced a new space and time fractional order based anisotropic diffusion equation for noise removal. It is a generalization of a method presented by Cuesta which incorporates between the heat and the wave equation by the use of time fractional derivatives, and the method proposed by Bai and Feng, which interpolates between the second and the fourth order anisotropic diffusion equation by the application of spatial fractional derivatives. This equation has the advantage of both of these methods. For the establishment of a numerical scheme, the proposed partial differential equation has been regarded as a spatially discretized fractional ordinary differential equation model by Caputo derivative. Currently the topic of variable order fractional calculus is developing in to more and more interest. In [5], the authors have constructed wavelet denoising models including variable order Caputo derivatives for removing noisy signals. Here the denoised signal was filtered by polynomials in a sequence of overlapped sub intervals. In [18], the authors have introduced fractional order model according to Caputo definition. The missing areas are occupied by using the model with fractional differentials of 8 directions.

2.4. Riesz fractional derivative. Riesz inaugurated n-dimensional integral potential operators to get explicitly the potential for hyperbolic, elliptic and parabolic Cauchy problems. The most interesting is the following weak singular integral operator \( I_\alpha \) defined by Riesz. Fractional integral of order \( \alpha \) in the Riesz
sense, denoted by, \( I^\alpha \) which is also known by Riesz potential and is defined by the Fourier convolution product

\[
I^\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x - \xi) f(\xi) d\xi I^\alpha \xi, \mathcal{R}(\alpha) > 0,
\]

\[
K_\alpha(x) = \frac{1}{\gamma_n(\alpha)} \begin{cases} 
\|x\|^{a-n}, & \text{where } a - n \neq 0, 2, ..., \\
\|x\|^{a-n} \log \left( \frac{1}{\|x\|} \right), & \text{where } a - n \neq 0, 2, ...
\end{cases}
\]

is the Riesz kernel. The normalized constant \( \gamma_n(\alpha) \) is given by

\[
\gamma_n(\alpha) = \frac{\pi^{n/2} 2^n \Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-\alpha}{2} \right)}.
\]

Riesz fractional derivative of \( f(x) \) in the Riesz sense, with \( x \in \mathbb{R}^n \) defined for \( \mathcal{R}(\alpha) > 0 \) by means of

\[
D^\alpha f(x) = \frac{1}{d_n(l, \alpha)} \int_{\mathbb{R}^n} \left( \frac{\Delta^2 f}{|\xi|^{n+\alpha}} \right)(x) d\xi,
\]

where \( l > \alpha \).

We can write this derivative in terms of convolution product and Fourier transform. Riez fractional derivative has verified for Leibniz rule. This derivative is mainly given as valid for order alpha equals 1, acts no differently than the other definition given in terms of its Fourier transform. In view of, we discuss the alpha goes to 1 limit of the space fractional quantum mechanics and its stability. In [21] authors derived an important fractional algorithm, variable order fractional centered difference based on the second order Riesz fractional differential operator. This model may be useful in clinical diagnosis and monitoring. It gives higher level texture enhancement of medical images. In [22] to solve the weakness of total variation based model for image restoration in this paper authors have analyzed and developed four algorithms for solving variation problem based on Split Bregman concept and discretize optimization problem by Riesz and Riemann fractional operator. Recently in [11] authors highlight the idea of fractional calculus and proposed a new mathematical model in using the convolution of fractional Tsallis entropy with the Riesz fractional derivative for image denoising. The structures of \( n \times n \) fractional mask windows in the \( x \) and \( y \) directions of this algorithm are constructed.
2.5. **Erdelyi-kober fractional derivative (E-K).** Erdelyi-kober fractional derivative is a generalization of the Riemann-Liouville fractional derivative, if often used, too. It is introduced by Arthur Erdelyi and Hemann Kober in 1940. The Erdelyi fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{\eta}^{\gamma, \delta} f(t) = \frac{\eta^{t-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_{a}^{x} \frac{S^{(n\gamma+\eta-1)}f(S)}{(tn-s^{\eta})^{1-\delta}} ds.$$  

This fractional integral operators are found to be quite useful in obtaining the solutions for single, dual and triple integral equations possessing special functions of mathematical and physics in their kernels.

We consider the right and left sided Erdelyi-kober fractional derivatives of order $\delta$ and $\alpha$ respectively. Let $n - 1 < \delta \leq n$, and $n \in \mathbb{N}$ and $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. The Erdelyi-kober fractional derivative is defined as

$$(2.4) \quad D_{\beta}^{\gamma, \delta} f(x) = \prod_{j=1}^{n} \left( \gamma + j + \frac{1}{\beta} x \frac{d}{dx} \right) \left( I_{\beta}^{\gamma+\delta,n-\delta} f \right)(x),$$

and

$$(2.5) \quad D_{\beta}^{\gamma, \alpha} f(x) = \prod_{0}^{m-1} \left( \gamma + j - \frac{1}{\beta} x \frac{d}{dx} \right) \left( I_{\beta}^{\gamma+\alpha,m-\alpha} f \right)(x).$$

But, Image denoising using this derivative is not in practice.

3. **Conclusion**

In this paper, we have discussed various fractional derivatives for denoising algorithms and their performance metrics are compared with individually. The adaptability of fractional derivative shows very good results in image denoising. Though the applications are different, the various numerical schemes based on the fractional derivative perform within their limit. There must be a technique which can be applied globally for all types of noisy images irrespective of the applications. It is expected that the future research gives the scope for such denoising algorithm based on Erdelyi-kober and Reisz which also helps in preserving the necessary sharp details of the image.
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