COMPUTABLE APPROXIMATION OF INTUITIONISTIC FUZZY TURING MACHINES

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ABSTRACT. First we define a new class of fuzzy Turing machine that we call as intuitionistic fuzzy Turing machine. This intuitionistic fuzzy Turing machine uses $t$-norm for membership degree, and its dual $t$-conorm for non-membership degree of inputs. We study the relation between recursive enumerable (r.e) languages and intuitionistic fuzzy Turing machines that accepts special type of intuitionistic fuzzy languages and also discuss the 2-recursive enumerable languages.

1. INTRODUCTION

The nondeterministic Turing machine after reading a input symbol it heads moves more than two states. It accepts a word if any sequence of moves reaches final states. But nondeterministic Turing machine is more powerful than deterministic Turing machine [3].

The Turing machine after reading a input symbol from input tape it moves either left or right side replacing the input symbol. The Turing machine accept a word after reaching final state [3].

There are number of models on classical computable approximation but most of them are equivalent. But fuzzy computability for those problems are different which depends upon the concepts of fuzzy computability. This models turn to

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In classical computability, the halting problem is not yet solved, but fuzzy computation solve halting problem and it was solved by Wiedermann [10, 11] and he finds FTM accept r.e and co-r.e languages.

FTMs and their languages are discussed by Zadeh [12], Lee [4] and santos [8]. Another generalization of fuzzy set is intuitionistic fuzzy set (IFS) which was introduced by Atanassov [1]. This IFS described by two functions expressing the degree of belongingness and the degree of non-belongingness.

Morteza Moniri [6] introduced Atanassov’s intuitionistic fuzzy languages and a new version of FTM called IFTM, for studying their computability theoretic properties. Morteza Moniri [6] use and the t-norm (∗) and the t-conorm (∗′) for calculating the degree of membership and non-membership for each path.

In section 2, we recall the definitions of t-norm, t-conorm, intuitionistic fuzzy Turing machine, recursive enumerable languages, intuitionistic fuzzy language accepted by a IFTM and also collect a definition of another way of fuzzy sets. In section 3 we define the recursive enumerable languages, co-recursive enumerable languages, and also prove that for any recursively enumerable language T, there is an intuitionistic fuzzy Turing machine M which accepts the intuitionistic fuzzy language \( F_T((a_1, a_2), (a_3, a_4)) \). In section 4 we define the 2-r.e languages and we prove that for any 2-recursive enumerable languages T their is an IFTM M which accepts intuitionistic fuzzification of 2-r.e languages T.

2. Preliminaries

Definition 2.1. [2] For any language \( A \) and we define the fuzzy set \( A : F_A(c, d) = \{(w, c) : w \in A\} \cup \{(w, d) : w \notin A\} \) where \( c \) and \( d \) are rationals \( (c, d \in I) \).

Definition 2.2. [9] A t-norm is a function \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) which satisfies the following properties:

- (T1) \( T(l, 1) = l \), for any \( l \in [0, 1] \);
- (T2) \( T(l, m) = T(m, l) \), for all \( l, m \in [0, 1] \);
- (T3) \( T(l, m) \leq T(l, n) \), for any \( l, m, n \in [0, 1] \) with \( m \leq n \);
- (T4) \( T(l, T(m, n)) = T(T(l, m), n) \) for any \( l, m, n \in [0, 1] \).

Definition 2.3. [9] A t-conorm is a function \( S : [0, 1] \times [0, 1] \rightarrow [0, 1] \) which satisfies the following properties:
(S1) \( S(l, 0) = l \), for any \( l \in [0, 1] \);
(S2) \( S(l, m) = S(m, l) \), for all \( l, m \in [0, 1] \);
(S3) \( S(l, m) \leq S(l, n) \), for any \( l, m, n \in [0, 1] \) with \( m \leq n \);
(S4) \( S(l, S(m, n)) = S(S(l, m), n) \) for any \( l, m, n \in [0, 1] \).

**Definition 2.4.** [7] A nondeterministic intuitionistic fuzzy Turing machine NIFTM is defined by

- \( Q \) is a finite set of states,
- \( \sum \) is the input alphabets,
- \( \Gamma \) is a finite set of tape symbols,
- \( \Delta \) is an intuitionistic fuzzy subsets of \( Q \times \Gamma \times 2^{Q \times \Gamma \times \{L,R\}} \) and \( \mu, \gamma : Q \times \Gamma \times 2^{Q \times \Gamma \times \{L,R\}} \to [0,1] \) are functions where \( \mu(\delta) + \gamma(\delta) \leq 1 \) \( \forall \delta \in \Delta \),
- \( q_0 \in Q \) is the starting state,
- \( \square \in \Gamma \) is a blank symbol,
- \( F \subseteq Q \) is the set of accepting states,
- \( * \) is a computable t-norm and \( *' \) is its computable \( t \)-conorm.

**Definition 2.5.** [7] An intuitionistic fuzzy language accepted by a nondeterministic intuitionistic fuzzy Turing machine (NIFTM) is defined by

\[ L(M) = \{(w, (\mu, \gamma)) | w \in \sum^*, (q_0 w, (1, 0)) \downarrow^* (u'q_f v', (\mu_t, \gamma_t)), q_f \in F, u', v' \in \Gamma^*\} \]

**Definition 2.6.** [5] A language \( L \) is said to be recursively enumerable (re) if there exists a TM that accepts it.

### 3. Intuitionistic Fuzzy Turing Machines as Acceptors

**Definition 3.1.** For any language \( T \subseteq \sum^* \), we define \( T(u) = 1 \) if \( u \in T \) and \( T(u) = 0 \) otherwise, a language \( T \subseteq \sum^* \) is recursive enumerable when there is a computable approximation \( f : \sum^* \times N \to \{0, 1\} \) such that

\[
\lim_{t \to \infty} f(u, t) = T(u),
\]

\[
(\forall u \in \sum^*) f(u, 0) = 0,
\]

\[
\#\{t : f(u, t) \neq f(u, t + 1)\} \leq 1,
\]
condition (3.1) says the approximation \( f \) stabilizes in 1 or 0 depends up on \( u \in T \) or \( u \notin T \). Condition (3.2) its starts from 0 and condition (3.3) for any \( u \in \sum^* \), \( f(u,t) \) can change one time.

First we define another way of intuitionistic fuzzy languages (IFL). Consider the language \( T \) and for rationals \( a_1, a_2, a_3 \) and \( a_4 \) (since \( a_1, a_2, a_3, a_4 \in [0,1] \)). We define the following intuitionistic fuzzification of the language \( T \)

\[
F_T((a_1, a_2), (a_3, a_4)) = \{ u, (a_1, a_2) : u \in T \} \cup \{ u, (a_3, a_4) : u \notin T \},
\]

where \( 0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1 \).

**Definition 3.2.** For any language \( T \subseteq \sum^* \), we define \( T(u) = 1 \) if \( u \in T \) and \( T(u) = 0 \) otherwise. \( T \subseteq \sum^* \) is co-re when there is a computable approximation \( f : \sum^* \times N \rightarrow \{0,1\} \) such that

\[
(3.4) \quad \lim_{t \to \infty} f(u,t) = T(u),
\]

\[
(3.5) \quad (\forall u \in \sum^*) f(u,0) = 1,
\]

\[
(3.6) \quad \#\{ t : f(u,t) \neq f(u,t+1) \} \leq 1,
\]

condition (3.4) says the approximation \( f \) stabilizes in 0 or 1 depends up on \( u \in T \) or \( u \notin T \). Condition (3.5) its starts from 1 and condition (3.6) for any word \( u \in \sum^* \) \( f(u,t) \) can change one time.

We define the contra positive version of intuitionistic fuzzification of the language \( T \). \( F_T((a_3, a_4), (a_1, a_2)) = \{ u, (a_3, a_4) : u \in T \} \cup \{ u, (a_1, a_2) : u \notin T \} \) where \( a_1, a_2, a_3 \) and \( a_4 \) are rationals \( a_1, a_2, a_3, a_4 \in [0,1] \) and \( 0 < a_3 + a_4 \leq 1, 0 < a_1 + a_2 \leq 1 \)

**Theorem 3.1.** Let \( T \subseteq \sum^* \) be any language and let \( a_1, a_2, a_3 \) and \( a_4 \) are rationals such that \( a_1 > a_3, a_2 < a_4 \) and \( 0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1 \). The following are equivalent

(i) \( T \) is r.e;

(ii) there is some IFTM which accepts an intuitionistic fuzzy language \( F_T((a_1, a_2), (a_3, a_4)) \).
Proof.

(i) ⇒ (ii) Assume that T is r.e and let \( f : \sum^* \times N \rightarrow \{0, 1\} \) be the approximation of T. Let M be an intuitionistic fuzzy Turing machine. After reading a word \( u \) it starts from \( q_0 \) and has two branches

- M goes from \( q_0 \) to \( q_f \) with intuitionistic fuzzy degree \((a_3, a_4)\); and

- M passes from the initial state \( q_0 \) to a process which scans \( f(u, 0), f(u, 1), f(u, 2), \ldots \) up to time \( t \) \( \ni f(u, t) = 1 \) (all process having with intuitionistic fuzzy degree \((1, 0)\)). If it happens then M reaches \( q_f \) with intuitionistic fuzzy degree \((a_1, a_2)\) and otherwise does not goes to \( q_f \).

If \( u \in T \) then there is a least \( s \ni f(u, s) = 1 \), so there will be two branches in M; the first branch with intuitionistic fuzzy degree \((a_3, a_4)\) and the second branch with intuitionistic fuzzy degree \((a_1, a_2)\) since \( a_1 > a_3, a_2 < a_4 \). Hence \( (u, (a_1, a_2)) \in L(M) \). On the other hand if \( u \notin T \) then the execution of M contains only one accepting path from the first branch. Hence \((u, (a_3, a_4)) \in L(M)\). Therefore IFTM which accepts the IFL \( F_T((a_1, a_2), (a_3, a_4)) \)

(ii) ⇐ (i) Suppose the intuitionistic fuzzy Turing machine M which accepts the intuitionistic fuzzy language \( F_T((a_1, a_2), (a_3, a_4)) \). we define \( f : \sum^* \times N \rightarrow \{0, 1\} \) as follows \( f(u, t) = 1 \), if time \( t \) we find that M reaches to \( q_f \) with intuitionistic fuzzy degree \((a_1, a_2)\). Otherwise \( f(u, t) = 0 \). Hence f is computable. Therefore T is r.e.

Corollary 3.1. Let \( T \subseteq \sum^* \) be a language and let \( a_1, a_2, a_3 \) and \( a_4 \) are rationals such that \( a_3 < a_1, a_4 > a_2 \) and \( 0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1 \).

(i) \( T \) is contra positive recursive enumerable;

(ii) there is some IFTM which accepts the IFL \( F_T((a_3, a_4), (a_1, a_2)) \).

Proposition 3.1. Let \( T \subseteq \sum^* \) be a non recursive r.e language and let \( a_1, a_2, a_3 \) and \( a_4 \) are rationals such that \( a_3 < a_1, a_4 > a_2 \) and \( 0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1 \) then the language \( F_T((a_3, a_4), (a_1, a_2)) \) is not accepted by any IFTM.

Proof. Let T be non recursive r.e language and assume that there is an IFTM M which accepts \( F_T((a_3, a_4), (a_1, a_2)) = \{u, (a_3, a_4) : u \in T\} \cup \{u, (a_1, a_2) : u \notin T\} \).

We check that whether there is an enumeration process for T. We consider the execution path of M after reading a input word \( u \). Then we search that either \( u \in T \) or search that accepting branch of M with intuitionistic fuzzy degree
(a_1, a_2)). Since a_3 < a_1, a_4 > a_2. Then we find an accepting branch with intuitionistic fuzzy degree (a_1, a_2). This is an enumeration process for T. This is a contradiction to our assumption that the language F_T((a_3, a_4), (a_1, a_2)) is accepted by any IFTM.

Corollary 3.2. Let T \subseteq \Sigma^* be a non recursive co-r.e language and a_1, a_2, a_3 and a_4 are rationals such that a_1 > a_3, a_2 < a_4 and 0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1 then the intuitionistic fuzzy language F_T((a_1, a_2), (a_3, a_4)) is not accepted by any IFTM.

Theorem 3.2. The following are equivalent:

(i) T is r.e.;

(ii) there is some rationals (r_1, r_2) \in (0, 1], 0 < r_1 + r_2 \leq 1 and some IFTM M such that \mu(u) > r_1, \gamma(u) < r_2 if and only if u \in T.

Proof.

(i) \Rightarrow (ii) Consider that T is recursive enumerable then there exists f : \Sigma^* \times N \rightarrow \{0, 1\}(computable approximation) \ni \# \{t : f(u, t) /\in f(u, t+1)\} \leq 2, \lim_{t \to \infty} f(u, t) =

Definition 4.1. Consider T is 2-recursive enumerable, then there is a f : \Sigma^* \times N \rightarrow \{0, 1\}(computable approximation) \ni #\{t : f(u, t) /\in f(u, t+1)\} \leq 2, \lim_{t \to \infty} f(u, t) =
$T(u) \text{ and } T(u, 0) = 0 \forall u$. In otherwise, $T(u, t)$ starts from 0, it change to 1 and may be go back to 0, when increasing $t$.

Theorem 4.1. The following are equivalent:

(i) $T$ is 2-recursive enumerable;
(ii) for any rationals $a_1, a_2, a_3$ and $a_4, a_1 > a_3 \geq 0, 0 \leq a_2 < a_4$ and $0 < a_1 + a_2 \leq 1, 0 < a_3 + a_4 \leq 1$ there is some IFTM $M$ such that $u \in T$ if and only if $\mu(u) \in (a_1, a_3)$ and $\gamma(u) \in (a_2, a_4)$.

Proof.

(i) $\Rightarrow$ (ii) Consider $T$ is 2-r.e and let $f : \Sigma^* \times \mathbb{N} \rightarrow \{0, 1\}$ be the approximation of $T$. Let $M$ be an IFTM. Assume that $u \in L(M)$, its initiate from $q_0$ and makes the three branches

- with intuitionistic fuzzy degree $(a_3, a_4)$, $M$ goes to accepting $q_f$.
- with intuitionistic fuzzy degree $(\frac{a_1 + a_3}{2}, \frac{a_2 + a_4}{2})$, $M$ goes to a process which finds the least stage time $t$ such that $f(u, t) = 1$, then it goes to $q_f$. If never happens, otherwise $M$ does not goes to $q_f$.
- with intuitionistic fuzzy degree $(a_1, a_2)$, $M$ goes to a process which finds least $t$ and $s$ such that $t < s$ and $f(w, t) = 1$ and $f(w, s) = 0$, it goes to $q_f$. If never happens, otherwise $M$ does not goes to $q_f$.

Now, suppose $u \in T$. There is a least time $t \ni f(u, t) = 1$. The approximation of two conditions, we have that for all $s \geq t f(u, s) = 1$. Then there is no accepting branch through the $3^{rd}$ branch. There are two branches the first one with intuitionistic fuzzy $(a_3, a_4)$, and the second with intuitionistic fuzzy $(\frac{a_1 + a_3}{2}, \frac{a_2 + a_4}{2}) > 0$. Hence $(u, (\frac{a_1 + a_3}{2}, \frac{a_2 + a_4}{2})) \in L(M)$. Suppose $u \notin T$ there are two ways (i) $f(u, t)$ does not change (ii) it changes twice, the first branch with intuitionistic fuzzy degree $(a_3, a_4)$. Hence $(u, (a_3, a_4)) \in L(M)$, in the up coming case there are three accepting branches, but one with maximum degree that is $3^{rd}$ branch, so $(u, (a_1, a_2)) \in L(M)$. Therefore $\mu(u) \in (a_1, a_3)$ and $\gamma(u) \in (a_2, a_4)$ if and only if $u \in T$.

(ii) $\Rightarrow$ (i)

Suppose $\mu(u) \in (a_1, a_3)$ and $\gamma(u) \in (a_2, a_4)$ if and only if $u \in T$. Let us discuss all the possibilities of $L(M)$: Define $f(u, 0) = 0$ and

$$f(u, t + 1) =$$
\[
\begin{cases}
0 \text{ if time } t, \text{ all accepting branch of } L(M) \text{ have membership degree } \\
\leq a_3, \text{ and non-membership degree } \geq a_4; \\
1 \text{ if time } t, \text{ there is an accepting branch of } L(M) \text{ with membership degree lies between } (a_3, a_4) \text{ and non-membership degree lies between } (a_1, a_2) \text{ and no accepting branch with membership degree } \\
\geq a_1 \text{ and non-membership degree } \leq a_2; \\
0 \text{ if time } t, \text{ there is an accepting branch of } L(M) \text{ with membership degree } \\
\geq a_1 \text{ and non-membership degree } \leq a_2.
\end{cases}
\]

Clearly, the computable approximation \( f(u, t) \) changes at most two times, when \( t \to \infty \). Hence \( T \) is 2-r.e. If \( u \in T \) at time \( t \), we search an accepting branch of \( M \) with accepting membership degree lies between \( (a_3, a_4) \) and non-membership degree lies between \( (a_1, a_2) \) and there cannot be any accepting path of membership degree \( \geq a_1 \) and non-membership degree lies between \( \leq a_2 \). Then for all \( s \geq t \) \( f(u, s) = 1 \). Otherwise, if \( u \notin T \) then either all accepting branches of \( L(M) \) have membership degree \( \leq a_3 \) and non-membership degree \( \geq a_4 \) or there is some accepting branch of membership degree \( \geq a_1 \) and non-membership degree \( \leq a_2 \); in both cases we consider that there is some \( t \) such that for all \( s \geq t \) \( f(u, s) = 0 \). Therefore \( T \) is 2-r.e. \( \square \)

5. Conclusion

The results presented in this paper are part of a continuing research on the foundations of intuitionistic fuzzy Turing machines. This paper introduces a special way of fuzzifying intuitionistic recursive enumerable languages into intuitionistic fuzzy languages. We proved that for any r.e language \( T \), there is a IFTM \( M \) which accepts the IFL \( F_T((a_1, a_2), (a_3, a_4)) \) and we proved that for any 2-recursive enumerable languages \( T \) their is an IFTM \( M \) which accepts intuitionistic fuzzification of 2-r.e languages \( T \).

References


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