ON SOME TYPES OF FUZZY ROUGH $\hat{P}G$–KERNEL SETS IN FUZZY ROUGH TOPOLOGICAL SPACE

S. PADMAPRIYA$^1$ AND V. NAVEENA

ABSTRACT. Using the notion of fuzzy sets. In this paper we define the concept of fuzzy rough closed sets and the expectation of a fuzzy rough $\hat{P}g$–closed sets. The new definition of expectation of fuzzy rough $\hat{P}g$–kernal sets in a fuzzy rough topological space. We derive some properties of these new concepts. By considering a weak fuzzy rough separation axioms like fuzzy rough $\hat{P}g – R_i$–spaces, $i = 0, 1, 2, 3$.

1. INTRODUCTION

The first publications in fuzzy set theory by Zadeh[1965] [6]. Thereafter Chang in [1968] [1] paved the way for the subsequent tremendous growth of numerous fuzzy topological concept. Fuzziness has so far not been defined uniquely semetically and probably never will be. It will mean different things, depending on the application area and the way it is measured. Z.Pawlak [3] introduced the concepts of rough sets in [1982]. Z.Pawlak introduced the concept of approximation given by the equivalence relation $R$ and the approximation space may not, in general be replaced by a membership function similar to that introduced by Zadeh. The concept of fuzzy rough topological spaces was introduced by S.Padmapriya, M.K.Uma and E.Roja [4]. In 1997, fuzzy generalized closed set was introduced by G.Balasubramania and P.Sundaram [5].

$^1$corresponding author

2010 Mathematics Subject Classification. 03E72, 94D05.

Key words and phrases. Fuzzy rough closed sets, fuzzy rough $\hat{P}g$–closed sets, fuzzy rough $\hat{P}g$–kernal sets, fuzzy rough $\hat{P}g – R_i$ spaces, $i = 0, 1, 2, 3$.
In this paper, we introduced the concept of weakly ultra fuzzy rough separation of two fuzzy rough sets in fuzzy rough topological space using fuzzy rough $\hat{p}g-$ closed sets. By these notion, we obtain that the kernel of a set in fuzzy rough topological space $(X, \tau)$ is a union of the set itself with the set of all boundary kernelled fuzzy point. Using this concept to define the fuzzy rough $\hat{p}g-$ kernal set of a fuzzy set $A$ of a fuzzy rough topological space $(X, \tau)$. We investigate some of the properties of weak fuzzy rough separation fuzzy rough $\hat{p}g-R_i-$ space, $i = 0, 1, 2, 3$.

2. Preliminaries

Fuzzy sets theory, introduced by L.A.Zadeh in 1965 [5], is the extension of classical set theory by allowing the membership of elements to range from 0 to 1. Let $X$ be the universe of a classical set of objects. Membership in a classical subset $A$ of $X$ is often viewed as a characteristic function $\mu_A$ from $X$ into $\{0, 1\}$, where

$$\mu_A(x) = \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \notin A \end{cases}$$

for any $x \in X$.

$\{0, 1\}$ is called a valuation set. If the valuation set is allowed to be the real interval $[0, 1]$, $A$ is called a fuzzy set in $X$. $\mu_A(x)$ (or simply $A(x)$) is the membership value (or degree of membership) of $x$ in $A$. Clearly, $A$ is a subset of $X$ that has no sharp boundary. A fuzzy set $A$ in $X$ can be represented by the set of pairs: $A = \{(x, A(x)), x \in X\}$.

Let $A : X \to [0, 1]$ be a fuzzy rough set. If $A(x) = 1$, for each $x \in X$, we denote it by $1_x$ and if $A(x) = 0$, for each $x \in X$, we denote it by $0_x$. That is, by $0_x$ and $1_x$, we mean the constant fuzzy rough sets taking the values 0 and 1 on $X$, respectively [2]. Let $I = [0, 1]$. The set of all fuzzy rough sets in $X$, denoted by $1^x$.

Throughout this paper, the fuzzy rough closure and the fuzzy interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively.

**Definition 2.1.** Let $A$ be a fuzzy rough set of a set $X$. The support of $A$ is the elements $x$ whose membership value is greater than 0, i.e., $\text{supp}(A) = \{x \in X : A(x) > 0\}$.
Definition 2.2. Let $A$ and $B$ be any two fuzzy rough sets in $X$. Then we define $A \lor B : X \to [0, 1]$ as follows: $(A \lor B)(x) = \max\{A(x), B(x)\}$. Also, we define $A \land B : X \to [0, 1]$ as follows: $(A \land B)(x) = \min\{A(x), B(x)\}$. By $A \lor B(A \land B)$, we mean the union (intersection) between two fuzzy rough sets $A$ and $B$ of $X$.

Definition 2.3. Let $A$ be any fuzzy rough set in a set $X$. The complement of $A$, is denoted by $1_x - A$ or $A^C$ and defined as follows: $A^C(x) = 1 - A(x)$, for each $x \in X$.

Remark 2.1. From definition 2.2 and definition 2.3, we have, if $A, B \in I^x$, then $A \lor B, A \land B$ and $1_x - A \in I^x$.

Definition 2.4. A fuzzy point $x_\lambda$ in a set $X$ is a fuzzy rough set defined as follows:

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \lambda \leq 1$. Now, $\text{supp}(x_\lambda) = \{y : x_\lambda(y) > 0\}$, but

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x \\ 0, & \text{otherwise,} \end{cases}$$

and $0 < \lambda \leq 1$. Then $\text{supp}(x_\lambda) = x$, so the value at $x$ is $\lambda$, and call the point $x$ its support of fuzzy point $x_\lambda$ and $\lambda$ is the height of $x_\lambda$. That is, $x_\lambda$ has the membership degree 0 for all $y \in X$ except one, say $x \in X$.

Definition 2.5. A fuzzy rough topology on a set $X$ is a family $\tau$ of fuzzy rough sets in $X$ which satisfies the following conditions:

(i) $0_x, 1_x \in \tau$,
(ii) If $A, B \in \tau$, then $A \land B \in \tau$,
(iii) If $\{A_i : i \in J\}$ is a family in $\tau$, then $i \in J A_i \in \tau$.

$\tau$ is called a fuzzy rough topology for $X$ and the pair $(X, \tau)$ (or simply $X$) is a fuzzy rough topological space or fts for short. Every element of $\tau$ is called $\tau$–fuzzy rough open set (fuzzy rough open set, for short). A fuzzy rough set is $\tau$–fuzzy rough closed (or simply fuzzy rough closed), if its complement is fuzzy rough open set. As ordinary topologies, the indiscrete fuzzy rough topology on $X$ contains only $0_x$ and $1_x$ (i.e., $\phi, X$), while the discrete fuzzy rough topology on $X$ contains all fuzzy sets in $X$.
3. Fuzzy Rough $\hat{p}g$—Closed Sets

Definition 3.1. A fuzzy rough subset $A$ of a fuzzy rough topological space $(X, \tau)$ is called a fuzzy rough $p-$open set (briefly $FRp-$ open set) if $A = \text{int}(\text{cl}(\text{int}(A)))$ and a fuzzy rough $p-$closed set (briefly $FRp-$ closed set) if $\text{cl}(\text{int}(A)) \leq A$.

Definition 3.2. A fuzzy rough subset $A$ of a fuzzy rough topological space $(X, \tau)$ is called a fuzzy rough generalized closed set (briefly $FRg-$ closed set) if $\text{cl}(A) \leq U$ whenever $A \leq U$ and $U$ is fuzzy rough open in $X$.

Definition 3.3. A fuzzy rough subset $A$ of a fuzzy rough topological space $(X, \tau)$ is called a fuzzy rough $pg-$closed set (briefly $FRpg-$ closed set) if $\text{pcl}(A) \leq U$ whenever $A \leq U$ and $U$ is fuzzy rough $p-$open in $X$.

Remark 3.1. Every fuzzy rough open (resp. fuzzy rough closed) set is a fuzzy rough $p-$open set (resp. fuzzy rough $p-$closed) set.

Proposition 3.1. In a fuzzy rough topological space $(X, \tau)$ the following hold and the converse of each statement is not true:

(i) Every fuzzy rough closed set is $FRg-$ closed.
(ii) Every fuzzy rough $p-$ closed set is $FRpg-$ closed.

Definition 3.4. A fuzzy rough subset $A$ of a fuzzy rough topological space $(X, \tau)$ is called fuzzy rough $\hat{p}-$generalized closed set (briefly $FR\hat{p}g-$ closed set) if $\text{int}(\text{cl}(\text{int}(A))) \leq U$ whenever $A \leq U$ and $U$ is fuzzy rough open in $X$. The complement of fuzzy rough $\hat{p}g-$ closed set in $X$ is fuzzy rough $\hat{p}g-$ open in $X$, the family of all fuzzy rough $\hat{p}g-$ open (fuzzy rough $\hat{p}g-$ closed) sets of an fuzzy rough topological space $(X, \tau)$ is denoted by $FR\hat{p}g-O(X)(FR\hat{p}g-C(X))$.

Definition 3.5. The intersection of all fuzzy rough $\hat{p}g-$ closed sets in $X$ containing $A$ is called fuzzy rough $\hat{p}-$ generalized closure of $A$ and is denoted by $\hat{p}g-\text{cl}(A)$, $\hat{p}g-\text{cl}(A) = \wedge\{B : A \leq B, B \text{ is fuzzy rough } \hat{p}g-\text{closed }\}$.

Theorem 3.1. If a fuzzy rough subset $A$ of $X$ is fuzzy rough $\hat{p}g$ closed in $X$, then $\text{int}(\text{cl}(A))$ does not contain any empty fuzzy rough open subset of $X$.

Proof. Let $A$ be fuzzy rough $\hat{p}g-$ closed. Let $U$ be fuzzy rough open set such that $\text{int}(\text{cl}(\text{int}(A))) - A \geq U$ and $U \neq 0$. That is $A \leq 1_x - U, \text{int}(A) \leq \text{int}(1_x - U) \leq 1_x - U, cl(int(A)) \leq 1_x - U, cl(int(A)) \leq 1_x - A, U \leq 1_x - [\text{int}(cl(int(A))]].$
Also $U \leq \text{int}(\text{cl}(\text{int}(A))), U \leq (1_x - [\text{int}(\text{cl}(\text{int}(A)))] \wedge \text{int}(\text{cl}(\text{int}(A))) = 0_x$, a contradiction. This completes the proof.

**Theorem 3.2.** If $A$ is fuzzy rough $\hat{pg}$—closed and $A \leq B \leq \text{int}(\text{cl}(\text{int}(A)))$, then $B$ is fuzzy rough $\hat{pg}$—closed.

**Proof.** Let $B \leq U$, where $U$ is fuzzy rough open. Then $\text{int}(\text{cl}(\text{int}(A))) \leq U, B = \text{int}(\text{cl}(\text{int}(A)))$. That is $B = \text{cl}(\text{int}(A)), \text{int}(B) \leq B \leq \text{cl}(\text{int}(A)), \text{cl}(\text{int}(B)) \leq \text{cl}(\text{cl}(\text{int}(A))) = \text{cl}(\text{int}(A)), \text{int}(\text{cl}(\text{int}(B))) \leq \text{int}(\text{cl}(\text{int}(A))) \leq U$. Hence $B$ is fuzzy rough $\hat{pg}$—closed.

**Theorem 3.3.** If $A$ is both fuzzy rough open and $FRg$—closed in $X$, then it is fuzzy rough $\hat{pg}$—closed in $X$.

**Proof.** Let $A$ be fuzzy rough open and $FRg$—closed in $X$. Let $A \leq U$ and $U$ be fuzzy rough open in $X$. Now $A \leq A, \text{cl}(A) \leq A$ by assumption. That is $\text{cl}(A) \leq U, \text{int}(A) \leq \text{cl}(A)$. Thus $\text{cl}(\text{int}(A)) \leq \text{cl}(A), \text{int}(\text{cl}(\text{int}(A))) \leq \text{cl}(A) \leq U$. Hence $A$ is fuzzy rough $\hat{pg}$—closed in $X$.

4. Fuzzy Rough $\hat{pg}$—Kernel and Fuzzy Rough $\hat{pg}$—$Ri$—Spaces

**Definition 4.1.** The intersection of all fuzzy rough $\hat{pg}$—open subset of $X$ containing $A$ is called the fuzzy rough $\hat{pg}$—kernel of $A$ (briefly $\hat{pg} - \ker(A)$), this means $\hat{pg} - \ker(A) \wedge \{G \in F\hat{pg} - O(X) : A \leq G\}$.

**Definition 4.2.** In a fuzzy rough topological space $(X, \tau)$, a fuzzy set $A$ is said to be weakly ultra fuzzy rough $\hat{pg}$—separated from $B$ if there exists a fuzzy rough $\hat{pg}$—open set $G$ such that $G \wedge B = 0_x$ or $A \wedge \hat{pg} - \text{cl}(B) = 0_x$.

By definition 4.2, we have the following: For every two distinct fuzzy points $x_\lambda$ and $y_\alpha$ of $X$,

(i) $\hat{pg} - \text{cl}(\{x_\lambda\}) = \{y_\alpha : \{y_\alpha\} \text{ is not weakly ultra fuzzy rough } \hat{pg} \text{—separated from } \{x_\lambda\}\}.$

(ii) $\hat{pg} - \ker(\{x_\lambda\}) = \{y_\alpha : \{x_\lambda\} \text{ is not weakly ultra fuzzy rough } \hat{pg} \text{—separated from } \{y_\alpha\}\}.$

**Corollary 4.1.** Let $(X, \tau)$ be a fuzzy rough topological space, then $y_\alpha \in \hat{pg} - \ker(\{x_\lambda\})$ iff $x_\lambda \in \hat{pg} - \text{cl}(\{y_\alpha\})$ for each $x \neq y \in X$. 
Proof. Suppose that \( y_\alpha \notin \hat{pg} - \ker(\{x_\lambda\}) \). Then there exists a fuzzy rough \( \hat{pg} - \) open set \( U \) containing \( x_\lambda \) such that \( y_\alpha \notin U \). Therefore, we have \( x_\lambda \notin \hat{pg} - \cl(\{y_\alpha\}) \).

The converse part can be proved in a similar way. \( \square \)

**Definition 4.3.** A fuzzy rough topological space \((X, \tau)\) is called fuzzy rough \( R_0\)-space \((FR\hat{pg} - R_0\)-space, for short\) if for each fuzzy rough \( \hat{pg}\)-open set \( U \) and \( x_\lambda \in U \), then \( \hat{pg} - \cl(\{x_\lambda\}) \leq U \).

**Definition 4.4.** A fuzzy rough topological space \((X, \tau)\) is called fuzzy rough \( R_1\)-space \((FR\hat{pg} - R_1\)-space, for short\) if for each two distinct fuzzy points \( x_\lambda \) and \( y_\alpha \) of \( X \) with \( \hat{pg} - \cl(\{x_\lambda\}) \neq \hat{pg} - \cl(\{y_\alpha\}) \), there exist disjoint fuzzy rough \( \hat{pg}\)-open sets \( U, V \) such that \( \hat{pg} - \cl(\{x_\lambda\}) \leq U \) and \( \hat{pg} - \cl(\{y_\alpha\}) \leq V \).

**Theorem 4.1.** Let \((X, \tau)\) be a fuzzy rough topological space. Then \((X, \tau)\) is \( FR\hat{pg} - R_0\)-space if and only if \( \hat{pg} - \cl(\{x_\lambda\}) = \hat{pg} - \ker(\{x_\lambda\}) \), for each \( x \in X \).

Proof. Let \((X, \tau)\) be a \( FR\hat{pg} - R_0\)-space. If \( \hat{pg} - \cl(\{x_\lambda\}) \neq \hat{pg} - \ker(\{x_\lambda\}) \), for each \( x \in X \), then there exist another fuzzy point \( y \neq x \) such that \( y_\alpha \in \hat{pg} - \cl(\{x_\lambda\}) \) and \( y_\alpha \notin \hat{pg} - \ker(\{x_\lambda\}) \); this means there exist an \( U_\alpha \) fuzzy rough \( \hat{pg}\)-open set, \( y_\alpha \notin U_\alpha \) implies \( \hat{pg} - \cl(\{x_\lambda\}) \leq U_\alpha \), this contradiction. Thus \( \hat{pg} - \cl(\{x_\lambda\}) = \hat{pg} - \ker(\{x_\lambda\}) \).

Conversely, let \( \hat{pg} - \cl(\{x_\lambda\}) = \hat{pg} - \ker(\{x_\lambda\}) \), for each fuzzy rough \( \hat{pg}\)-open set \( U, x_\lambda \in U \), then \( \hat{pg} - \ker(\{x_\lambda\}) = \hat{pg} - \cl(\{x_\lambda\}) \leq U \) [By definition 4.1]. Hence by definition 4.3, \((X, \tau)\) is a \( FR\hat{pg} - R_0\)-space. \( \square \)

**Theorem 4.2.** A fuzzy rough topological space \((X, \tau)\) is an \( FR\hat{pg} - R_0\)-space if and only if for each \( G \) fuzzy rough \( \hat{pg}\)-closed set and \( x_\lambda \in G \), then \( \hat{pg} - \ker(\{x_\lambda\}) \leq G \).

Proof. Let for each \( G \) fuzzy rough \( \hat{pg}\)-closed set and \( x_\lambda \in G \), then \( \hat{pg} - \ker(\{x_\lambda\}) \leq G \) and let \( U \) be a fuzzy rough \( \hat{pg}\)-open set, \( x_\lambda \in U \) then for each \( y_\alpha \in U \) implies \( y_\alpha \in U^c \) is a fuzzy rough \( \hat{pg}\)-closed set implies \( \hat{pg} - \ker(\{y_\alpha\}) \leq U^c \) [By assumption]. Therefore \( x_\lambda \notin \hat{pg} - \ker(\{y_\alpha\}) \) implies \( y_\alpha \notin \hat{pg} - \cl(\{x_\lambda\}) \) [By corollary 4.1]. So \( \hat{pg} - \cl(\{x_\lambda\}) \leq U \). Thus \((X, \tau)\) is an \( FR\hat{pg} - R_0\)-space.

Conversely, let a fuzzy rough topological space \((X, \tau)\) be \( FR\hat{pg} - R_0\)-space and \( G \) be fuzzy rough \( \hat{pg}\)-closed set and \( x_\lambda \in G \). Then for each \( x_\lambda \notin G \) implies \( y_\alpha \in G^c \) is fuzzy rough \( \hat{pg}\)-open set, then \( \hat{pg} - \cl(\{y_\alpha\}) \leq G^c \) [Since \((X, \tau)\) is \( \hat{pg} - R_0\)-space], so \( \hat{pg} - \ker(\{x_\lambda\}) = \hat{pg} - \cl(\{x_\lambda\}) \). Thus \( \hat{pg} - \ker(\{x_\lambda\}) \leq G \). \( \square \)
Corollary 4.2. A fuzzy rough topological space $(X, \tau)$ is $FR^{\hat{p}g} - R_0$-space if and only if for each $U$ fuzzy rough $\hat{p}g-$ open set and $x_\lambda \in U$, then $\hat{p}g - cl(\hat{p}g - ker(\{x_\lambda\})) \leq U$.

Proof. Clearly. \qed

Theorem 4.3. Every $FR^{\hat{p}g} - R_1$-space is a $FR^{\hat{p}g} - R_0$-space.

Proof. Let $(X, \tau)$ be a $FR^{\hat{p}g} - R_1$-space and let $U$ be a fuzzy rough $\hat{p}g-$ open set, $x_\lambda \in U$, then for each $y_\alpha \notin U$ implies $y_\alpha \in U^C$ is an fuzzy rough $\hat{p}g-$ closed set and $\hat{p}g - cl(\{y_\alpha\}) \leq U^C$ implies $\hat{p}g - cl(\{x_\lambda\}) \neq \hat{p}g - cl(\{y_\alpha\})$. Hence by definition 4.4, $\hat{p}g - cl(\{x_\lambda\}) \leq U$. Thus $(X, \tau)$ is a $FR^{\hat{p}g} - R_0$-space. \qed

Theorem 4.4. A fuzzy rough topological space $(X, \tau)$ is $FR^{\hat{p}g} - R_1$-space if and only if for each $x \neq y \in X$ with $\hat{p}g - ker(\{x_\lambda\}) \neq \hat{p}g - (\{y_\alpha\})$, then there exist fuzzy rough $\hat{p}g-$ closed sets $G_1, G_2$ such that $\hat{p}g - ker(\{x_\lambda\}) \leq G_1$, $\hat{p}g - ker(\{x_\lambda\}) \wedge G_2 = 0_x$ and $\hat{p}g - ker(\{y_\alpha\}) \leq G_2$, $\hat{p}g - ker(\{y_\alpha\}) \wedge G_2 = 0_x$, and $G_1 \cup G_2 = 1_x$.

Proof. Let a fuzzy rough topological space $(X, \tau)$ be $FR^{\hat{p}g} - R_1$-space. Then for each $x \neq y \in X$ with $\hat{p}g - ker(\{x_\lambda\}) \neq \hat{p}g - (\{y_\alpha\})$. Since every $FR^{\hat{p}g} - R_1$-space is a $FR^{\hat{p}g} - R_0$-space [by theorem 4.3], and by theorem 4.1, $\hat{p}g - cl(\{x_\lambda\}) \neq \hat{p}g - cl(\{y_\alpha\})$, then there exist fuzzy rough $\hat{p}g-$ open sets $U_1, U_2$ such that $\hat{p}g - cl(\{x_\lambda\}) \leq U_1$ and $\hat{p}g - cl(\{y_\alpha\}) \leq U_2$ and $U_1 \cup U_2 = 0_x$ [since $(X, \tau)$ is $FR^{\hat{p}g} - R_1$-space], then $U_1^C$ and $U_2^C$ are fuzzy rough $\hat{p}g-$ closed sets such that $U_1^C \cup U_2^C = 1_x$. Put $G_1 = U_1^C$ and $G_2 = U_2^C$. Thus $x_\lambda \in U_1 \leq G_2$ and $y_\alpha \in U_2 \leq G_1$ so that $\hat{p}g - ker(\{x_\lambda\}) \leq U_1 \leq G_2$ and $\hat{p}g - ker(\{y_\alpha\}) \leq U_2 \leq G_1$.

Conversely, let for each $x \neq y \in X$ with $\hat{p}g - ker(\{x_\lambda\}) \neq \hat{p}g - ker(\{y_\alpha\})$, there exist fuzzy rough $\hat{p}g-$ closed sets $G_1, G_2$ such that $\hat{p}g - ker(\{x_\lambda\}) \leq G_1$, $\hat{p}g - ker(\{x_\lambda\}) \wedge G_2 = 0_x$ and $\hat{p}g - ker(\{y_\alpha\}) \leq G_2$, $\hat{p}g - ker(\{y_\alpha\}) \wedge G_2 = 0_x$, and $G_1 \vee G_2 = 1_x$, then $G_1^C$ and $G_2^C$ are fuzzy rough $\hat{p}g-$ open sets such that $G_1^C \wedge G_2^C = 0_x$. Put $G_1^C = U_2$ and $G_2^C = U_1$. Thus, $\hat{p}g - ker(\{x_\lambda\}) \leq U_1$ and $\hat{p}g - ker(\{y_\alpha\}) \leq U_2$ and $U_1 \cup U_2 = 0_x$, so that $x_\lambda \in U_1$ and $y_\alpha \in U_2$ implies $x_\lambda \notin \hat{p}g - cl(\{y_\alpha\})$ and $y_\alpha \notin \hat{p}g - cl(\{x_\lambda\})$, then $\hat{p}g - cl(\{x_\lambda\}) \leq U_1$ and $\hat{p}g - cl(\{y_\alpha\}) \leq U_1$. Thus, $(X, \tau)$ is a $FR^{\hat{p}g} - R_1$-space. \qed

Corollary 4.3. A fuzzy rough topological space $(X, \tau)$ is $\hat{p}g - R_1$-space if and only if for each $x \neq y \in X$ with $\hat{p}g - cl(\{x_\lambda\}) \neq \hat{p}g - cl(\{y_\alpha\})$ there exist disjoint fuzzy $\hat{p}g-$ open sets $U, V$ such that $\hat{p}g - cl(\hat{p}g - ker(\{x_\lambda\})) \leq U$ and $\hat{p}g - cl(\hat{p}g - ker(\{y_\alpha\})) \leq V$. 


Proof. Let \((X, \tau)\) be a \(FR\mathring{p}g - R_1\) space and let \(x \neq y \in X\) with \(\mathring{p}g - cl(\{x\}) \neq \mathring{p}g - cl(\{y\})\), then there exist disjoint fuzzy rough \(\mathring{p}g\) open sets \(U, V\) such that \(\mathring{p}g - cl(\{x\}) \leq U\) and \(\mathring{p}g - cl(\{y\}) \leq V\). Also \((X, \tau)\) is \(FR\mathring{p}g - R_0\) space [by theorem 4.3] implies for each \(x \in X\), then \(\mathring{p}g - cl(\{x\}) = \mathring{p}g - ker(\{x\})\) [By theorem 4.1], but \(\mathring{p}g - cl(\{x\}) = \mathring{p}g - cl(\mathring{p}g - cl(\{x\})) = \mathring{p}g - cl(\alpha g - ker(\{x\}))\).

Thus \(\mathring{p}g - cl(\mathring{p}g - ker(\{x\})) \leq U\) and \(\mathring{p}g - cl(\mathring{p}g - ker(\{y\})) \leq V\). Conversely, let for each \(x \neq y \in X\) with \(\mathring{p}g - cl(\{x\}) \neq \mathring{p}g - cl(\{y\})\) there exist disjoint fuzzy rough \(\mathring{p}g\) open sets \(U, V\) such that \(\mathring{p}g - cl(\{x\}) \leq U\) and \(\mathring{p}g - cl(\{y\}) \leq V\). Since \(\{x\}\mathring{p}g - ker(\{x\})\), then \(\mathring{p}g - cl(\{x\}) \leq \mathring{p}g - cl(\mathring{p}g - ker(\{x\}))\) for each \(x \in X\). So we get \(\mathring{p}g - cl(\{x\}) \leq U\) and \(\mathring{p}g - cl(\{y\}) \leq V\). Thus, \((X, \tau)\) is a \(FR\mathring{p}g - R_1\) space.

\(\Box\)

Definition 4.5. Let \((X, \tau)\) be a fuzzy rough topological space. Then \(X\) is called:

(i) fuzzy rough \(\mathring{p}g\) regular space (\(FR\mathring{p}gr\) space, for short), if for each fuzzy point \(x_\lambda\) and each fuzzy rough \(\mathring{p}g\) closed set \(F\) such that \(x_\lambda \in 1_x - F\), there exist disjoint fuzzy rough \(\mathring{p}g\) open sets \(U\) and \(V\) such that \(x_\lambda \in U\) and \(F \leq V\).

(ii) fuzzy rough \(\mathring{p}g\) normal space (\(FR\mathring{p}gn\) space, for short) iff for each pair of disjoint fuzzy rough \(\mathring{p}g\) closed sets \(A\) and \(B\), there exist disjoint fuzzy rough \(\mathring{p}g\) open sets \(U\) and \(V\) such that \(A \leq U\) and \(B \leq V\).

(iii) fuzzy rough \(\mathring{p}g - R_2\) space (\(FR\mathring{p}g - R_2\) space, for short) if it is property fuzzy rough \(\mathring{p}g\) regular space.

(iv) fuzzy rough \(\mathring{p}g - R_3\) space (\(FR\mathring{p}g - R_3\) space, for short) iff it is \(FR\mathring{p}g - R_1\) space and \(FR\mathring{p}gn\) space.

Remark 4.1. Every \(FR\mathring{p}g - R_k\) space is a \(FR\mathring{p}g - R_{k-1}\) space, \(k = 2, 3\).

Proof. Clearly. \(\Box\)

Theorem 4.5. A fuzzy rough topological space \((X, \tau)\) is \(FR\mathring{p}gr\) space (\(FR\mathring{p}g - R_2\) space) if and only if for each fuzzy rough \(\mathring{p}g\) closed subset \(G\) of \(X\) and \(x_\lambda \notin G\) with \(\mathring{p}g - ker(G) \neq \mathring{p}g - ker(\{x_\lambda\})\) then there exist fuzzy rough \(\mathring{p}g\) closed sets \(F_1, F_2\) such that \(\mathring{p}g - ker(G) \leq F_1, \mathring{p}g - ker(G) \land F_2 = 0_x\) and \(\mathring{p}g - ker(\{x_\lambda\}) \leq F_1, \mathring{p}g - ker(\{x_\lambda\}) \land F_2 = 0_x\) and \(F_1 \lor F_2 = 1_x\).

Proof. Let a fuzzy rough topological space \((X, \tau)\) be \(FR\mathring{p}gr\) space (\(FR\mathring{p}g - R_2\) space) and let \(G\) be a fuzzy rough \(\mathring{p}g\) closed set, \(x_\lambda \notin G\), then there exist
disjoint fuzzy rough $\hat{pg}$—open sets $U, V$ such that $G \subseteq V, x_\lambda \in V$ and $U \land V = 0_x$, then $U^C$ and $V^C$ are fuzzy rough $\hat{pg}$—closed sets such that $U^C \land V^C = 1_x$. Put $F_2 = U^C$ and $F_1 = V^C$, so we get $\hat{pg} - ker(G) \subseteq U \subseteq F_1, \hat{pg} - ker(G) \land F_2 = 0_x$ and $\hat{pg} - ker(\{x_\lambda\}) \subseteq V \subseteq F_2, \hat{pg} - ker(\{x_\lambda\}) \land F_1 = 1_x$ and $F_1 \lor F_2 = 1_x$.

Conversely, let for each fuzzy rough $\hat{pg}$—closed subset $G$ of $X$ and $x_\lambda \in G$ with $\hat{pg} - ker(G) \neq \hat{pg} - ker(\{x_\lambda\})$, then there exist fuzzy rough $\hat{pg}$—closed sets $F_1, F_2$ such that $\hat{pg} - ker(G) \subseteq F_1, \hat{pg} - ker(G) \land F_2 = 0_x$ and $\hat{pg} - ker(\{x_\lambda\}) \subseteq F_2, \hat{pg} - ker(\{x_\lambda\}) \land F_2 = 0_x$ and $F_1 \lor F_2 = 1_x$. Then $F_1^C$ and $F_2^C$ are fuzzy rough $\hat{pg}$—opensets such that $F_1^C \land F_2^C = 0_x$ and $\hat{pg} - ker(\{x_\lambda\}) \land F_1^C = 0_x$, $\hat{pg} - ker(\{x_\lambda\}) \land F_2^C = 0_x$. So that $G \subseteq F_1$ and $x_\lambda \in F_1^C$. Thus, $(X, \tau)$ is $FR\hat{pg}$—space ($F\hat{pg} - R_2$ - space).

**Lemma 4.1.** Let $(X, \tau)$ be a $FR\hat{pg}$—space and $F$ be a fuzzy rough $\hat{pg}$—closed set. Then $\hat{pg} - ker(F) = F = \hat{pg} - cl(F)$.

**Proof.** Let $(X, \tau)$ be a $FR\hat{pg}$—space and $F$ be a fuzzy rough $\hat{pg}$—closed set. Then for each $x_\lambda \notin F$, there exist disjoint fuzzy rough $\hat{pg}$—open sets $U, V$ such that $F \subseteq U$ and $x_\lambda \in V$. Since $\hat{pg} - ker(F) \subseteq U$, implies $\hat{pg} - ker(F) \land V = 0_x$, thus $x_\lambda \notin \hat{pg} - cl(\hat{pg} - ker(F))$. We showing that if $x_\lambda \notin F$ implies $x_\lambda \notin \hat{pg} - cl(\hat{pg} - ker(F))$, therefore $\hat{pg} - cl(\hat{pg} - ker(F)) \subseteq F = \hat{pg} - cl(F)$. As $\hat{pg} - cl(F) = F \subseteq \hat{pg} - ker(F)$ [By definition 4.1]. Thus, $\hat{pg} - ker(F) = F = \hat{pg} - cl(F)$. □

**Theorem 4.6.** A fuzzy rough topological space $(X, \tau)$ is $FR\hat{pg}$—space ($FR\hat{pg} - R_2$ - space) iff for each fuzzy rough $\hat{pg}$—closed subset $F$ of $X$ and $x_\lambda \notin F$ with $\hat{pg} - cl(\hat{pg} - ker(F)) \neq \hat{pg} - cl(\hat{pg} - ker(\{x_\lambda\}))$, then there exist disjoint fuzzy rough $\hat{pg}$—open sets $U, V$ such that $\hat{pg} - cl(\hat{pg} - ker(F)) \subseteq U$ and $\hat{pg} - cl(\hat{pg} - ker(\{x_\lambda\})) \subseteq V$.

**Proof.** Let a fuzzy rough topological space $(X, \tau)$ is $FR\hat{pg}$—space ($FR\hat{pg} - R_2$ - space) and let $F$ be a fuzzy rough $\hat{pg}$—closed set, $x_\lambda \notin F$. Then there exist disjoint fuzzy rough $\hat{pg}$—open set $U, V$ such that $F \subseteq U$ and $x_\lambda \in V$. By lemma 4.1, $\hat{pg} - cl(\hat{pg} - ker(F)) = \hat{pg} - cl(F) = F$, in the other hand $(X, \tau)$ is a $FR\hat{pg} - R_0$-space [By theorem 4.3 and remark 4.1]. Hence, by theorem 4.1, $\hat{pg} - ker(\{x_\lambda\}) = \hat{pg} - ker(\{x_\lambda\})$, for each $x \in X$. Thus, $\hat{pg} - cl(\hat{pg} - ker(F)) \subseteq U$ and $\hat{pg} - cl(\hat{pg} - ker(F)) \subseteq V$.

Conversely, let for each fuzzy $\hat{pg}$—closed set $F$ and $x_\lambda \notin F$ with $\hat{pg} - cl(\hat{pg} - ker(F)) \neq \hat{pg} - cl(\hat{pg} - ker(\{x_\lambda\}))$, then there exist disjoint fuzzy rough $\hat{pg}$—open
sets $U,V$ such that $\hat{pg} – cl(\hat{pg} – ker(F)) \leq U$ and $\hat{pg} – cl(\hat{pg} – ker(\{x_\lambda\})) \leq V$. Then $F \leq V$ and $x_\lambda \in V$. Thus, $(X,\tau)$ is $FR\hat{pg}r – space (FR\hat{pg} – R_2$-space).

**Theorem 4.7.** A fuzzy rough topological space $(X,\tau)$ is $FR\hat{pg}n$-space if and only if for each disjoint fuzzy rough $\hat{pg}$-closed sets $G,H$ with $\hat{pg} – ker(G) \neq \hat{pg} – ker(H)$ then there exist fuzzy rough $\hat{pg}$-closed sets $F_1,F_2$ such that $\hat{pg} – ker(G) \leq F_1$, $\hat{pg} – ker(G) \wedge F_2 = 0_x$ and $\hat{pg} – ker(H) \leq F_2$, $\hat{pg} – ker(H) \leq F_1 = 0_x$ and $F_1 \cup F_2 = 1_x$.

**Proof.** Let a fuzzy rough topological space $(X,\tau)$ be $FR\hat{pg}n$-space and let for each disjoint fuzzy rough $\hat{pg}$-closed sets $G,H$ with $\hat{pg} – ker(G) \neq \hat{pg} – ker(H)$ then there exist disjoint fuzzy rough $\hat{pg}$-open sets $U,V$ such that $G \leq U$ and $H \leq V$ and $U \wedge V = 0_x$, then $U^C$ and $V^C$ are fuzzy rough $\hat{pg}$-closed sets such that $U^C \cup V^C = 1_x$ and $\hat{pg} – ker(G) \wedge U^C = 0_x$, $\hat{pg} – ker(H) \wedge V^C = 0_x$. Put $U^C = F_2$ and $V^C = F_1$. Thus, $\hat{pg} – ker(G) \leq F_1$, $\hat{pg} – ker(G) \wedge F_2 = 0_x$ and $\hat{pg} – ker(H) \leq F_2$, $\hat{pg} – ker(H) \leq F_1 = 0_x$.

Conversely, let for each disjoint fuzzy rough $\hat{pg}$-closed sets $G,H$ with $\hat{pg} – ker(G)\hat{pg} – ker(H)$, there exist fuzzy rough $\hat{pg}$-closed sets $F_1,F_2$ such that $\hat{pg} – ker(G) \leq F_1$, $\hat{pg} – ker(G) \wedge F_2 = 0_x$ and $\hat{pg} – ker(H) \leq F_2$, $\hat{pg} – ker(H) \wedge F_1 = 0_x$ and $F_1 \wedge F_2 = 1_x$ implies $F_1^C$ and $F_2^C$ are fuzzy rough $\hat{pg}$-open sets such that $F_1^C \wedge F_2^C = 0_x$. Put $F_1^C = V$ and $F_2^C = U$, thus $\hat{pg} – ker(G) \leq U$ and $\hat{pg} – ker(H) \leq (V), so that G \leq U and G \leq V. Thus (X,\tau)$ is $FR\hat{pg}n$-space.

**Theorem 4.8.** Every $FR\hat{pg} – R_3$-space is $FP\hat{gr}$-space.

**Proof.** Let $F$ be a fuzzy rough $\hat{pg}$-closed and $x_\lambda \notin F$. Then $\hat{pg} – ker(\{x_\lambda\}) \neq \hat{pg} – ker(F)$, then for each $x_\lambda \in F$ there exist fuzzy rough $\hat{pg}$-closed sets $G_{y_\alpha},H_{y_\alpha}$ such that $\hat{pg} – ker(\{y_\alpha\}) \leq G_{y_\alpha}$, $\hat{pg} – ker(\{y_\alpha\}) \wedge H_{y_\alpha} = 0_x$ and $\hat{pg} – ker(\{x_\lambda\}) \leq H_{y_\alpha}$, $\hat{pg} – ker(\{x_\lambda\}) \wedge G_{y_\alpha} = 0_x$ [since $(X,\tau)$ is $FR\hat{pg} – R_1$-space and by theorem 4.4], let $\beta \wedge H_{y_\alpha} : x_\lambda \in H_{y_\alpha}$, so we have $\beta \wedge F = 0_x$. Hence $(X,\tau)$ is $FR\hat{pg}n$-space, then there exist disjoint fuzzy rough $\hat{pg}$-open sets $U,V$ such that $F \leq U$ and $x_\lambda \in \beta \leq V$. Thus, $(X,\tau)$ is $FR\hat{pg}r$-space.

**References**


PADMAVANI ARTS AND SCIENCE COLLEGE FOR WOMEN
SALEM, TAMIL NADU, INDIA
Email address: vnaveena14996@gmail.com

PADMAVANI ARTS AND SCIENCE COLLEGE FOR WOMEN
SALEM, TAMIL NADU, INDIA
Email address: priyasathi17@gmail.com