RECONSTRUCTION OF FINITE TOPOLOGICAL SPACES
WITH MORE THAN ONE ISOLATED POINT

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ABSTRACT. The deck of a topological space $X$ is the set $\mathcal{D}(X) = \{[X - \{x\}] : x \in X\}$, where $[Z]$ denotes the homeomorphism class of $Z$. A space $X$ is topologically reconstructible if whenever $\mathcal{D}(X) = \mathcal{D}(Y)$ then $X$ is homeomorphic to $Y$. For $|\mathcal{D}(X)| \geq 3$, it is shown that all finite topological spaces with more than one isolated point are reconstructible.

1. FIRST SECTION: IMPORTANT

A vertex-deleted subgraph or card $G-v$ of a graph $G$ is obtained by deleting the vertex $v$ and all edges incident with $v$. The collection of all cards of $G$ is called the deck of $G$. A graph $H$ is a reconstruction of $G$ if $H$ has the same deck as $G$. A graph is said to be reconstructible if it is isomorphic to all its reconstructions. A parameter $p$ defined on graphs is reconstructible if, for any graph $G$, it takes the same value on every reconstruction of $G$. The graph reconstruction conjecture, posed by Kelly and Ulam [7] in 1941, asserts that every graph $G$ on $n$ ($\geq 3$) vertices is reconstructible. More precisely, if $G$ and $H$ are finite graphs with at least three vertices such that $\mathcal{D}(H) = \mathcal{D}(G)$, then $G$ and $H$ are isomorphic.

In 2016, Pitz and Suabedissen [6] have introduced the concept of reconstruction in topological spaces as follows. For a topological space $X$, the subspace $X - \{x\}$ is called a card of $X$. The set $\mathcal{D}(X) = \{[X - \{x\}] : x \in X\}$ of

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subspaces of $X$ is called the deck of $X$, where $[X - \{x\}]$ denotes the homeomorphism class of the card $X - \{x\}$. Given topological spaces $X$ and $Z$, we say that $Z$ is a reconstruction of $X$ if their decks agree. A topological space $X$ is said to be reconstructible if the only reconstructions of it are the spaces homeomorphic to $X$. Formally, a space $X$ is reconstructible if $D(X) = D(Z)$ implies $X \cong Z$ and a property $\mathcal{P}$ of topological spaces is reconstructible if $D(X) = D(Z)$ implies "$X$ has $\mathcal{P}$ if and only if $Z$ has $\mathcal{P}$".

The number of elements in a topological space $X$ is called the size of $X$. Terms not defined here are taken as in [2]. Gartside et al [3, 4, 6] have proved that the space of real numbers, the space of rational numbers, the space of irrational numbers, every compact Hausdorff space that has a card with a maximal finite compactification, and every Hausdorff continuum $X$ with weight $\omega(X) < |X|$ are reconstructible. In their above paper, they also proved certain properties of a space, namely all hereditary separation axioms and all cardinal invariants are reconstructible. All finite sequences are reconstructed by Manvel et al [5].

In this paper, it is shown that every finite topological space with at least $n \geq 4$ elements and more than one isolated point with $|D(X)| \geq 3$ is reconstructible. Also, for $|D(X)| = 2$, we prove that the finite topological spaces with more than one isolated point and with one discrete card is reconstructible. The condition that $n \geq 4$ is needed because there are nonreconstructible topological spaces of size 2 or 3. For $n = 2$, the set $X = \{a, b\}$ endowed with any of the three topologies $\tau_1 = \{\phi, \{a\}, \{b\}, X\}, \tau_2 = \{\phi, \{a\}, X\}$ or $\tau_3 = \{\phi, X\}$ is not reconstructible, since all these topological spaces have the same deck. For $n = 3$, the set $X = \{a, b, c\}$ endowed with any of the two topologies $\tau_1 = \{\phi, \{c\}, X\}, \tau_2 = \{\phi, \{a, b\}, X\}$ is not reconstructible.

2. Finite Topological Spaces

Since every discrete topological space is reconstructible [1], we assume that $X$ is a finite topological space of size $n$, which is not discrete, where $n \geq 4$ and $X = \{x_1, x_2, \ldots, x_n\}$. Let $m = |D(X)|$. The next lemma is proved in [1].

Lemma 2.1. [1] Let $X$ be a topological space with isolated points and for $m = 2$, all but one isolated point in a card must belong to at least one open set in the other cards. Then the property that whether $X$ has one isolated point or at least two isolated points is reconstructible.
For a collection of open subsets \( Y \) of a topological space \( X \), \( \vee(Y) \) denotes the set consisting of elements of \( Y \) together with all possible union of elements of \( Y \).

**Theorem 2.1.** Every space with at least three mutually non-homeomorphic cards and more than one isolated point is reconstructible.

**Proof.** Let \( X \) be a space with more than one isolated point. Then every card of \( X \) has at least one isolated point. If an isolated point, say \( x_1 \) of a card \( X \), is not an isolated point of any other card of \( X \), then \( x_1 \) is not an isolated point of \( X \) (as otherwise, \( x_1 \) would be an isolated point in all but one card of \( X \)) and hence the set \( \{x_1, x\} \) is open in \( X \). Thus, an isolated point, say \( x_1 \) in a card \( X \), is an isolated point of \( X \) if it is isolated in all but one card of \( X \); \( \{x_1, x\} \) is open in \( X \) otherwise. Repeat these arguments for each of the remaining isolated points in every card in the deck of \( X \) to identify the isolated points of \( X \). Finally, we arrive at two disjoint new sets, say \( O_1 \) and \( O_2 \), where \( O_1 \) consists of all the isolated points of \( X \) and \( O_2 \) consists of all such open sets \( \{x_1, x\} \) of \( X \). Let \( O_1 = \{y_1, y_2, ..., y_k\} \), where \( k \geq 2 \) and let \( \mathcal{C}_1 = \vee(O_1 \cup O_2) \).

Now consider a card \( X_x \) and an open set \( U_2 = \{a, b\} \) of \( X_x \) such that \( U \notin \mathcal{C}_1 \). Then \( |U_2 \cap O_1| = 1 \) or 0. If the former holds, without loss of generality, let \( a = y_i \), for some \( i, 1 \leq i \leq k \). If \( b \) is not open in none of the cards, then \( \{a, b\} \) is not open in \( X \) (as otherwise \( U_2 \) would be an open set in \( n - 2 \) cards, \( \{b\} \) would be an open set in \( X - \{a\} \) and \( \{a\} \) would be an open set in \( X - \{b\} \), giving a contradiction) and hence the set \( U_2 \cup \{x\} \) is open in \( X \). If \( \{b\} \) is open in a card, which is not open in \( X \), then \( \{b\} \) along with the deleted point for the card, in which \( \{b\} \) is open, is open in \( X \). Note that this open set already in the collection \( O_2 \). So, assume that the latter holds. If one of \( \{a\} \) and \( \{b\} \) are not open in none of the cards, then \( U_2 \) is not open in \( X \) and hence the set \( U_2 \cup \{x\} \) is open in \( X \). So assume \( \{a\} \) and \( \{b\} \) are open in at least one of the cards. If one of \( \{a\} \) and \( \{b\} \) are not open in \( X \), then the 1-subset which is not open in \( X \) along with the deleted point for the card, in which the 1-subset is present, is open in \( X \). Therefore, \( U_2 \) is not open in \( X \) and hence \( U_2 \cup \{x\} \) is open in \( X \). Repeat these steps for each 2-open set \( W \) in every card in the deck of \( X \). Finally, we will get collections of open sets, \( O_3 \) consists of some 3-open sets of \( X \) that is not in \( \mathcal{C}_1 \). Let \( \mathcal{C}_2 = \vee(\mathcal{C}_1 \cup O_3) \).
Again we proceed with the similar arguments to 3-open sets. Consider any card $X_x$ and a 3-open set, say $U_3$ in $X_x$ such that $U_3 \notin \mathcal{C}_2$. If one of the 2-subsets of $U_3$ is not open in none of the cards, then $U_3$ is not open in $X$ (as otherwise $V$ would be open in $n - 3$ cards and 2-subsets of $U_3$ are open in the three cards $X_x$, where $z \in U_3$, giving a contradiction) and hence $U_3 \cup \{x\}$ is open in $X$. So assume that, all the 2-subsets of $U_3$ is open in at least one the cards. If one of the 2-subset is not open in $X$, then the 2-subset which is not open in $X$ along with deleted point for the corresponding card, in which the 2-subset is present, is open in $X$. Therefore, $U_3$ is not open in $X$ and hence $U_3 \cup \{x\}$ is open in $X$. Repeat these steps for each 3-open set $U_3$ in every card in the deck of $X$. Finally, we shall arrive at collections of open sets, say $O_4$ consists of some 4-open sets of $X$ that is not in $\mathcal{C}_2$. Let $\mathcal{C}_3 = \bigvee (\mathcal{C}_2 \cup O_4)$.

In general, consider a card $X_x$ and a $k$-open set, say $U_k$, $k \leq n - 2$, in $X_x$ such that $U_k \notin \mathcal{C}_{k-1}$. If one of the $(k - 1)$-subsets of $U_k$ is not open in none of the cards, then $U_k$ is not open in $X$ and hence $U_k \cup \{x\}$ is open in $X$. So assume that, all the $(k - 1)$-subsets of $U_k$ is open in at least one the cards. If one of the $(k - 1)$-subset is not open in $X$, then the $(k - 1)$-subset along with deleted point for the corresponding card, in which the $(k - 1)$-subset is present, is open in $X$. Therefore, $U_k$ is not open in $X$ and hence $U_k \cup \{x\}$ is open in $X$. Repeat these steps for each $k$-open set $U_k$ in every card in the deck of $X$. Finally, we shall arrive at collections of open sets, say $O_{k+1}$ consists of some $(k + 1)$-open sets of $X$ that is not in $\mathcal{C}_{k-1}$. Let $\mathcal{C}_k = \bigvee (\mathcal{C}_{k-1} \cup O_{k+1})$.

The proof completes once we identified the remaining $(n - 1)$-open sets, if any, in $X$. For this, we consider a card $X_x$ such that the unique $(n - 1)$-open set $X - \{x\}$ in it is not in the collection $\mathcal{C}_{n-2}$ so formed. Since each card has at least one isolated point of $X$, it follows that each $(n - 1)$-open set in a card contains at least one isolated point of $X$. Now, let $\mathcal{U}(X_x) = \{X_x - y_i : y_i \text{ is an isolated point of } X\}$. If an element of $\mathcal{U}(X_x)$ does not belong to any card, then $X_x$ does not open in $X$, since the element itself is not in the space $X$. So, assume that each element of $\mathcal{U}(X_x)$ is an open set of at least one card of $X$. If at least one of the elements of the set $\mathcal{U}(X_x)$ is open in $X$, then the set $X_x$ is open in $X$. So, assume that none of the elements of the set $\mathcal{U}(X_x)$ is open in $X$. Then each element in $\mathcal{U}(X_x)$ together with the deleted point of the card, in which the element is open, is open in $X$ and hence $X_x$ is not open in $X$. Repeat these steps for the $(n - 1)$-open set in every card in the deck of $X$. Let $O'_{n-1}$ be the set
of these new \((n-1)\)-open sets. Then \(\mathcal{G}_{n-2} \cup O'_{n-1}\) is the desired topology on \(X\).

\[\Box\]

**Lemma 2.2.** Let \(X\) be a space with only two non-homeomorphic cards and more than one isolated point. If the subspace topology on one card, say \(X_x\) is the discrete topology, then \(\tau_X\) must be equal to one of the following three collections:

(i) \(\tau_{X_x} \cup X\);

(ii) \(\tau_{X_x} \cup \{x, y\} \cup \{\{x, y\} \cup U \mid y \in X_x \text{ and } U \in \tau_{X_x}\}\);

(iii) \(\tau_{(X_x-y)} \cup \{x, y\} \cup \{\{x, y\} \cup U \mid y \in X_x \text{ and } U \in \tau_{(X_x-y)}\}\).

**Proof.** Assume to the contrary, that \(\tau_X\) was not equal to the collection given in (i), (ii) and (iii). We proceed by three cases depending on the number of isolated points of \(X\).

**Case 1.** The space \(X\) has \(n-1\) isolated points.

Let \(\{y_1, y_2, ..., y_{n-1}\}\) be the set of all isolated points of \(X\). By our contrary assumption, there exists a smallest \(i\)-open set, say \(W\) in \(X\) containing the point, say \(x \in X - \{y_i, ..., y_{n-1}\}\) for some \(i, 3 \leq i \leq n-1\). Then the only card having discrete topology is \(X_x\). Consider now the two cards \(X_{y_k}\) and \(X_{y_s}\), where \(y_k \in W\) and \(y_s \notin W\). We claim that the two cards \(X_{y_k}\) and \(X_{y_s}\) are non-homeomorphic.

Suppose, to the contrary, that there is a homeomorphism \(f : X_{y_k} \rightarrow X_{y_s}\). Then \(x\) must be mapped to \(x\) under \(f\). It is clear that the smallest open set containing the point \(x\) in \(X_{y_k}\) is \(W - \{y_k\}\) while the smallest open set containing the point \(x\) in \(X_{y_s}\) is \(W\), giving a contradiction to \(f\). This completes the claim and hence the space \(X\) has at least three mutually non-homeomorphic cards, giving a contradiction.

**Case 2.** The space \(X\) has \(n-2\) isolated points.

Let \(\{y_1, y_2, ..., y_{n-2}\}\) be the set of all isolated points of \(X\). By our contrary assumption, in \(X\), there exists an open set \(U \cup \{x, y\}\) or \(U \cup \{x\}\) or \(U \cup \{y\}\) where \(\phi \neq U \in \mathcal{V}((y_1, y_2, ..., y_{n-2}))\) and \(x, y \in X - \{y_1, y_2, ..., y_{n-2}\}\). If the former holds, then no card has the discrete topology, a contradiction. So, assume that the latter holds. If \(|U| > 1\), then no card will have the discrete topology, a contradiction. So, let us assume that \(|U| = 1\). If \(X\) has only one open set \(\{y_j\} \cup \{x\}\), where \(j = 1, 2, ..., n-2\), then no cards will have the discrete topology, again a contradiction. So, assume that \(X\) has two open sets \(\{y_i\} \cup \{x\}\) and \(\{y_j\} \cup \{y\}\), where \(i, j = 1, 2, ..., n-2\). If \(i \neq j\), then no card will have the discrete topology. Otherwise, the only card having the discrete topology is \(X_{y_i}\). Since the card \(X_{y_i}\),
The space \( X \) has at most \( n - 3 \) isolated points, where \( n \geq 5 \).

The isolated points of \( X \) are denoted by \( y_1, y_2, ..., y_n \). Then \( 2 \leq i \leq n - 3 \), since the space \( X \) under consideration has at least two isolated points. If \( X \) has no 2-open sets of the type \( \{ y_i, x_j \} \), where \( x_j \in X - \{ y_1, y_2, ..., y_i \}, 1 \leq j \leq n - i \), then no card has the discrete topology. So, assume that \( X \) has 2-open sets \( \{ y_i, x_j \} \). If \( X \) has at most \( k \), where \( k < n - i \), 2-open sets of the above type, then no card has the discrete topology. So, we assume that \( X \) has \( n - i \) 2-open sets of the form \( \{ y_i \} \cup \{ x_j \} \). If any two of the isolated points \( y_i \) are distinct, then clearly no card has the discrete topology. Finally, we consider the case that all the \( y_i \) are equal and they are \( y_a \). Now, the only card having the discrete topology is \( X_{y_a} \). Since the card \( X_{y_a} \) has \( i - 1 \) isolated points while the card \( X_{x_j} \) has \( i \) isolated points, it follows that they are non-homeomorphic, giving a contradiction and completes the proof. \( \square \)

**Theorem 2.2.** Let \( X \) be a space with only two non-homeomorphic cards and more than one isolated point. If one card has discrete topology, then \( X \) is reconstructible.

**Proof.** Let the two cards be \( X_x, X_y \), where \( X_x \) is endowed with discrete topology. By Lemma 2.2, \( \tau_X \) must be equal to one of the collections given in (i), (ii) or (iii). Therefore \( X_y \) has \( n - 2 \) or \( n - 3 \) isolated points. We proceed by two cases depending on the number of isolated points in the card \( X_y \).

**Case 1.** The card \( X_y \) has \( n - 2 \) isolated points.

Suppose \( X \) has \( n - 2 \) isolated points. Then \( X \) must be of the form (iii) of Lemma 2.2 and hence the card \( X_y \) has \( n - 3 \) isolated points, a contradiction. Hence \( X \) must contain exactly \( n - 1 \) isolated points and consequently \( \tau_X \) must be equal to the form (i) or (ii) of Lemma 2.2. Let the set of all isolated points in \( X \) be \( \{ y_1, y_2, ..., y_{n-1} \} \). If the open sets of \( X_y \) are in \( \bigvee \{ \{ y_1, y_2, ..., y_i \} \} \cup X_y \), where \( 2 \leq i \leq n - 2 \), then \( X \) has no open set of the form \( \{ y_i, x \} \), where \( i = 1, 2, ..., n - 1 \) (as otherwise \( X_y \) would contain the 2-open set \( \{ y_j, x \} \), where \( j = 1, 2, ..., n - 1 \)). Then, by Lemma 2.2, \( X \) must be of the form (i) of Lemma 2.2. Now the collection \( \{ U \mid U \in X_y \text{ and } U \in \bigvee \{ \{ y_1, y_2, ..., y_{n-2} \} \} \} \cup \{ U \cup \{ y \} \mid U \in X_y \} \) is the desired topology on \( X \). Suppose that \( X_y \) contains the open set of the form \( \{ y_i, x \} \). Then, by Lemma 2.2, \( X \) must be of the form (ii) of Lemma 2.2 and hence the collection \( \{ U \mid U \in X_y \} \cup \{ U \cup \{ y \} \mid U \in X_y \} \) is the desired topology on \( X \).
Case 2. The card $X_y$ has $n - 3$ isolated points.

Now $X$ has $n - 2$ isolated points. By Lemma 2.2, $\tau_X$ is of the form (iii) of Lemma 2.2. Since $X$ has $n - 2$ isolated points, one isolated point in the card $X_x$ is not open in $X$; let it be $y$. Then the collection $\{U \cup \{x\} \mid U \in X_x$ and $y \in U\} \cup \vee(\{y_1, y_2, ..., y_{n-2}\})$ is the desired topology on $X$.  \hfill \Box

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