COMPLEMENTARY NIL G-ECCENTRIC DOMINATION IN FUZZY GRAPHS

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ABSTRACT. A g-eccentric dominating set $D \subseteq V$ of a fuzzy graph $G = (\rho, \phi)$ is said to be a complementary nil g-eccentric dominating set (CNGED-set) if $V - D$ contains no g-eccentric dominating set of $G = (\rho, \phi)$. The least scalar cardinality taken over all CNGED- set of $G$ is called the complementary nil g-eccentric domination number of $G = (\rho, \phi)$. In this article, bounds for complementary nil g-eccentric domination number for a few standard fuzzy graph are given and theorems related to CNGED- sets are discussed. The relation between complementary nil g-eccentric domination number and other well known parameters are analyzed.

1. INTRODUCTION

Tamilchelvam and Robinson Chelladurai [10] defined the approach of complementary nil domination on graph in 2009. In 2020, Mohamed Ismayil and Muthupandiyan [7] presented the concept of g-eccentric domination in fuzzy graph. A fuzzy graph $G = (\rho, \phi)$ characterized with two functions $\rho$ characterized on $V$ and $\phi$ characterized on $E \subseteq V \times V$, where $\rho : V \rightarrow [0, 1]$ and $\phi : E \rightarrow [0, 1]$ such that $\phi(x, y) \leq \rho(x) \land \rho(y) \forall x, y \in V$. We expect that $V$ is finite and non-empty, $\phi$ is reflexive and symmetric. We indicate the crisp graph by $G^* = (\rho^*, \phi^*)$ of the fuzzy graph $G(\rho, \phi)$, where $\rho^* = \{x \in V : \rho(x) > 0\}$ and $\phi^* = \{(x, y) \in E : \phi(x, y) > 0\}$. The fuzzy graph $G = (\rho, \phi)$ is called trivial in

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this case $|\rho^*| = 1$. For all the definitions pertinent to fuzzy graph we bring up in [8,9,11].

A strong path $\pi$ from $r$ to $s$ is called geodesics in a fuzzy graph in the event that there is no shorter strong path from $r$ to $s$ and a length of a $r - s$ geodesic is the geodesic distance(g-distance) from $r$ to $s$ denoted by $d_g(r, s)$. The geodesic eccentricity (g-eccentricity) $e_g(x)$ of a node $x$ in a connected fuzzy graph $G = (\rho, \varphi)$ is characterized by $e_g(x) = \max\{d_g(x, y), y \in V\}$. The least g-eccentricity among the vertices of $G$ is called g-radius and indicated by $r_g(G) = \min\{e_g(x), x \in V\}$ and the greatest g-eccentricity among the vertices of $G$ is called g-diameter and indicated by $d_g(G) = \max\{e_g(x), x \in V\}$. A vertex $y$ is said to be a g-central node in case $e_g(y) = r_g(G)$. Moreover, a vertex $y$ in $G$ is said to be a g-peripheral node in case $e_g(y) = d_g(G)$. For all the definitions pertinent to g-eccentricity we bring up in [3,5–7].

A subset $D$ of $V$ is called a dominating set of $G$ in the event that for each $y \in V - D$ there exists $x \in D$ such that $x$ dominates $y$. The least cardinality taken over all the minimal dominating set is called the domination number of $G$ and it is indicated by $\gamma(G)$. A dominating set $D \subseteq V(G)$ in a fuzzy graph $G = (\rho, \varphi)$ is said to be a g-eccentric dominating set each vertex $y \notin D$, there exists at least one g-eccentric vertex $x \in D$. The least scalar cardinality taken over all g-eccentric dominating set is called g-eccentric domination number and is indicated by $\gamma_{ged}(G)$. For all the definitions pertinent to eccentric domination we bring up in [1,2,4,6,7].

In this article, the new domination parameter $\gamma_{cnged}(G)$ is presented. complementary nil g-eccentric domination set and its number for a few standard fuzzy graph are characterized and a few theorems related to complementary nil g-eccentric domination are expressed and demonstrated.

2. Complementary Nil G-Eccentric Domination in Fuzzy Graph

In this chapter, a modern g-eccentric dominating parameter known as complementary nil g-eccentric domination number is characterized. The relationship between $\gamma_{cnged}(G)$ and other well known parameters are obtained.

**Definition 2.1.** A g-eccentric dominating set $D \subseteq V(G)$ of a fuzzy graph $G(\rho, \varphi)$ is called complementary nil g-eccentric dominating set(CNGED -set) in the event that $V - D$ is not a g-eccentric dominating set. A CNGED-set $D$ is minimal if no proper
subset $D' \subset D$ be a CNGED-set. The Complementary nil $g$-eccentric domination number of $G$ is the least scalar cardinality taken over all the minimal CNGED-set and is indicated by $\gamma_{cnged}(G)$. The most prominent scalar cardinality taken over all the minimal CNGED-set is called an upper complementary nil $g$-eccentric domination number and is implied by $\Gamma_{cnged}(G)$.

**Example 1.** In the Figure 1, a minimum dominating set is $D_1 = \{x_3, x_5\}$, a minimum $g$-eccentric dominating set is $D_2 = \{x_3, x_5\}$, a minimum CNGED-set is $D_3 = \{x_3, x_5, x_6\}$ and minimal CNGED-sets are $D_4 = \{x_1, x_3, x_5, x_6\}$ and $D_5 = \{x_2, x_3, x_4, x_5, x_6\}$. The minimum complementary nil dominating set is $D_6 = \{x_3, x_4, x_5, x_6\}$. Therefore $\gamma(G) = 0.8$, $\gamma_{ged}(G) = 0.8$, $\gamma_{cnged}(G) = 1.2$, $\gamma_{cnd}(G) = 1.5$ and $\Gamma_{cnged}(G) = 2.2$. Here, $\gamma_{cnd}(G) > \gamma_{cnged}(G)$.

**Note 2.1.** In a fuzzy graph $G = (\rho, \phi)$, there is no relation between $\gamma_{cnged}(G)$ and $\gamma_{cnd}(G)$.

**Note 2.2.** The minimum CNGED-set is denoted by $\gamma_{cnged}(G)$.

**Observation 2.1.** For any fuzzy graph $G = (\rho, \phi)$

1. $\gamma(G) \leq \gamma_{ged}(G) \leq \gamma_{cnged}(G)$.
2. Every super set of a CNGED-set is also a CNGED-set.
3. Complement of a CNGED-set is not a CNGED-set.
4. Complement of a $\gamma$-set is need not a CNGED-set.
5. $\gamma_{cnged}$-set need not be unique.
6. $\gamma_{cnged}(G) \leq \Gamma_{cnged}(G)$.
7. Complete fuzzy graph has no CNGED-set.
Example 2. In the figure 2, we have the minimum dominating set is $D_1 = \{x_3, x_4\}$, minimum g-eccentric dominating set is $D_2 = \{x_3, x_4\}$, minimum complementary nil dominating set is $D_3 = \{x_2, x_3, x_4\}$ and minimum complementary nil g-eccentric dominating set is $D_4 = \{x_2, x_3, x_4\}$. Hence, $\gamma(G) = 0.5$, $\gamma_{ged}(G) = 0.5$, $\gamma_{cnd}(G) = 0.9$ and $\gamma_{cnged}(G) = 0.9$. Here, $\gamma_{cnd}(G) = \gamma_{cnged}(G)$.

Example 3. In the figure 3, we have the minimum dominating set is $D_1 = \{x_2, x_5\}$, minimum g-eccentric dominating set is $D_2 = \{x_3, x_4, x_6\}$, minimum complementary nil dominating set is $D_3 = \{x_2, x_3, x_4, x_5\}$ and minimum complementary nil g-eccentric dominating set is $D_4 = \{x_3, x_4, x_6\}$. Hence, $\gamma(G) = 0.5$, $\gamma_{ged}(G) = 1.5$, $\gamma_{cnd}(G) = 1.4$ and $\gamma_{cnged}(G) = 1.5$. Here, $\gamma_{cnd}(G) < \gamma_{cnged}(G)$. 
**Definition 2.2.** Let $S \subseteq V(G)$ in a fuzzy graph $G(\rho, \phi)$. A vertex $x \in S$ is said to be an enclave of $S$ if $\phi(x,y) < \rho(x) \wedge \rho(y)$ for all $x \in V - S$, that is $N_s[x] \subseteq S$.

**Definition 2.3.** Let $S \subseteq V$ be a subset in a fuzzy graph $G(\rho, \phi)$. A vertex $x \in S$ is said to be a g- eccentric enclave of $S$ if $E_g(x) \subseteq S$.

**Example 4.** In the fuzzy graph given in figure 3.1 a vertex $x_5$ is an enclave of $D_5$ and the vertex $x_3$ is g-eccentric enclave of $D_3$ since $E_g(x_3) = \{x_5, x_6\} \subseteq D_3$.

3. Theorems Related to Complementary Nil g-Eccentric Domination in Fuzzy Graph

In this chapter, some theorems related to complementary nil g-eccentric domination in fuzzy graphs are stated and proved.

**Theorem 3.1.** Let $G(\rho, \phi)$ be a fuzzy graph and $D \subseteq G$ be a CNGED-set. Then

(i) $D$ contains at least one enclave or
(ii) $D$ has at least one vertex $y$ such that $e_g(y) = d_g(x,y), \forall x \in D$.

**Proof.** Let $D$ be a CNGED - set of a fuzzy graph $G(\rho, \phi)$. By the definition of CNGED -set, $V - D$ is not a g-eccentric dominating set. At that point there exists a vertex $x \in D$ such that (i) $x$ is not strong neighbours to any of the vertices in $V - D$. That is $N_s[x] \subseteq D$. In this manner, $D$ contains at least one enclave or (ii) $x$ has no g-eccentric vertex in $V - D$, Hence $x$ has all its g-eccentric vertices are in $D$ only. \(\square\)

**Theorem 3.2.** Let $D$ be a CNGED-set of a fuzzy graph $G(\rho, \phi)$. At that point $D$ is minimal $\Leftrightarrow$ for each $x \in D$ one of the given conditions satisfied.

(i) $x$ contains no strong neighbors in $D$ or g-eccentric vertex of $u$ is not in $D$.
(ii) There exists a few $y \in V - D$ such that $N_u(y) \cap D = \{x\}$ or $E_g(y) \cap D = \{x\}$.
(iii) $[V - D] \cup \{x\}$ could be a g-eccentric dominating set.

**Proof.** Assume $D$ is a minimal CNGED-set. If there exists a vertex $x \in D$ such that $x$ does not fulfill any of the given conditions (i), (ii) and (iii). **Case(i):** Assume $x$ contains all strong neighbors at that point $D$ is not negligible. **Case (ii):** Suppose g-eccentric vertex of $x$ is in $D$ then $D$ is not minimal, by (ii) $N_s(y) \cap D = \varphi$, $D$ is not dominating set and $E_g(y) \cap D = \varphi$ at that point $D$ is not g-eccentric dominating set **Case(iii):** $[V - D] \cup \{x\}$ is not a g-eccentric
dominating set. This infers that \( D - \{x\} \) may be a CNGED-set of \( G(\rho, \phi) \), which is a contradiction to the minimality of \( D \).

Conversely, let \( D \) be a CNGED-set and for all \( x \in D \), one of the three conditions hold. We claim that \( D \) must be a minimal CNGED-set. Assume that \( D \) is not a minimal CNGED-set, then there exists a vertex \( x \in D \) such that \( D - \{x\} \) is a CNGED-set. Thus, \( x \) is strong neighbours to at least one vertex in \( D - \{x\} \) and \( x \) has g-eccentric vertex in \( D - \{x\} \), which implies (i) does not hold.

Moreover, if \( D - \{x\} \) is a CNGED-set, each vertex \( x \) in \( [V - D] \cup \{x\} \) is strong neighbors to at least one vertex in \( D - \{x\} \) and \( x \) has a g-eccentric vertex in \( D - \{x\} \). Subsequently condition (ii) does not hold. Since \( D - \{x\} \) is a CNGED-set, at that point by observation 2.1(3) \( [V - D] \cup \{x\} \) is not a CNGED-set, therefore condition (iii) does not hold.

Hence, there exists \( x \in D \) such that \( x \) does not fulfill the conditions (i), (ii) and (iii), which is a contradiction to our assumption.

**Theorem 3.3.** Let \( D \) be a CNGED-set of a connected fuzzy graph \( G(\rho, \phi) \), and g-eccentric point of \( x \) in \( D - \{x\} \), then there exists a vertex \( x \in D \) such that \( D - \{x\} \) is g-eccentric dominating set.

**Proof.** Let \( D \) be a CNGED-set of a fuzzy graph \( G(\rho, \phi) \). By a theorem 3.1, each CNGED-set has at least one enclave in \( D \). Let \( x \in D \) be an enclave of \( D \). Implies that \( \phi(x, y) < \rho(x) \land \rho(y) \) for all \( y \in V - D \), that is \( N_s[x] \subseteq D \). Since \( G(\rho, \phi) \) is a connected fuzzy graph, at that point there exists at least one vertex \( z \in D \) such that \( \phi(x, z) \leq \rho(x) \land \rho(z) \) and g-eccentric vertex of \( x \) is in \( D - \{x\} \). Hence \( D - \{x\} \) is a g-eccentric dominating set.

**Theorem 3.4.** A CNGED-set in a connected fuzzy graph \( G(\rho, \phi) \) is not singleton.

**Proof.** Let \( D \) be a CNGED-set of a connected fuzzy graph \( G(\rho, \phi) \). By a Theorem 3.1, each CNGED-set has at least one enclave in \( D \) or \( D \) has at least one vertex whose g-eccentric vertices in \( D \). Let \( x \in D \) be an enclave of \( D \). Implies that \( \phi(x, y) < \rho(x) \land \rho(y) \) for all \( y \in V - D \), that is

\[
N_s[x] \subseteq D.
\]

Suppose \( D \) contains only a vertex \( x \), (3.1) does not exists or isolated in \( G(\rho, \phi) \), which is a contradiction to our connectedness. Thus CNGED-set has more than one vertex in a connected fuzzy graph \( G(\rho, \phi) \).
Corollary 3.1. Let $D$ be a $\gamma_{cnged}$-set of a fuzzy graph $(\rho, \phi)$. If $x$ and $y$ are two enclave of $D$, then

(i) $N_s[x] \cap N_s[y] \neq \varnothing$ and

(ii) $x$ and $y$ are strong neighbors.

Example 5. In figure 1 $x_5, x_6$ are two enclave of $D_6$.

Theorem 3.5. A $\gamma_{cnged}$ set in a fuzzy graph $G(\rho, \phi)$ is not independent.

Proof. Let $G(\rho, \phi)$ be a fuzzy graph and $D$ be a CNGED-set which is independent. Then $D$ is a minimal g-eccentric dominating set which infers that $V - D$ is additionally a g-eccentric dominating set. By the definition, $D$ is not CNGED-set, which is a contradistinction. \hfill \Box

Observation 3.1. For any fuzzy graph $G(\rho, \phi)$, each $\gamma_{cnged}$-set intersects with each $\gamma_{ged}$-set of $G(\rho, \phi)$.

Theorem 3.6. In a star fuzzy graph $S_\rho$, $\gamma_{cnged}(K_{\rho_1, \rho_2}) = \rho_1 \ell + \rho_2 \ell$, where $\rho_1 \ell = \min\{\rho(x), x \in \rho_1\}$ and $\rho_2 \ell = \min\{\rho(y), y \in \rho_2\}$ and $|\rho_1^*| = 1$ and $|\rho_2^*| \geq 1$.

Proof. Let $K_{\rho_1, \rho_2}$ be a star fuzzy graph. Let $D = \{x, y\}$, where $x$ may be a central vertex which dominates all the vertices in $V - D$ and $y$ is pendent vertex, that is $y$ is the g-eccentric vertices of $V - D$, $V - D$ is not g-eccentric dominating set. In this manner, $D$ is $\gamma_{cnged}$-set. Clearly, g-eccentric dominating set is also CNGED-set. That is, $\gamma_{ged}(K_{\rho_1, \rho_2}) = \gamma_{cnged}(K_{\rho_1, \rho_2}) = \rho_1 \ell + \rho_2 \ell$. \hfill \Box

Theorem 3.7. Let $(K_{\rho_1, \rho_2})$ be a complete bipartite fuzzy graph, then $\gamma_{cnged}(K_{\rho_1, \rho_2}) \leq \min(|\rho_1|, |\rho_2|) + \rho_n$, where $\rho_n = \max\{\rho(x), x \in V\}$.

Proof. $(K_{\rho_1, \rho_2})$ be a complete bipartite fuzzy graph, $\rho = \rho_1 \cup \rho_2$ where $m = |\rho_1^*|$ and $n = |\rho_2^*|$ such that each vertex of $V_1$ is strong neighbors of a vertex in $V_2$ and vice versa. Let $D = \rho_1 \cup \{y\}, y \in V_2$. Since $V - D$ is not g-eccentric dominating set, at that point $D$ is CNGED-set.

Subsequently, $\gamma_{cnged}(K_{\rho_1, \rho_2}) \leq \min(|\rho_1|, |\rho_2|) + \rho_n$. \hfill \Box

Corollary 3.2.

(i) Let $(K_{\rho_1, \rho_2})$ be the complete bipartite fuzzy graph, then

$$\gamma_{cnged}(K_{\rho_1, \rho_2}) = \begin{cases} |\rho_1| + \rho_20, & \text{if } |\rho_1| \leq |\rho_2| \\ |\rho_2| + \rho_10, & \text{if } |\rho_1| > |\rho_2| \end{cases}$$
where \( \rho_{10} \) is the minimum membership value of \( \rho_1 \) and \( \rho_{20} \) are the minimum membership value of \( \rho_2 \).

(ii) Let \( T_\rho \) be a fuzzy tree. Then \( \gamma_{cnged}(T_\rho) \leq \gamma(T_\rho) + \rho_n \).

**Corollary 3.3.** In a fuzzy wheel graph \( W_\rho \), \( |\rho^*| = 4 \) has no CNGED-set.

**Proof.** By observation 2.1(7), complete fuzzy graph has no CNGED-set. Hence, \( W_\rho, |\rho^*| = 4 \) is a complete fuzzy graph with 4 vertices has no CNGED-set. \( \square \)

4. Bounds for Complementary Nil g-Eccentric Domination in Fuzzy Graph

In this chapter, we talk about theorem related to bounds for few fuzzy standard graphs.

**Observation 4.1.** For any fuzzy graph \( G = (\rho, \phi) \)

(1) \( 2\rho_0 \leq \gamma_{cnged} \leq p - \rho_0 \).

**Observation 4.2.**

(1) For a complete fuzzy graph \( K_\rho, \gamma_{cnged}(K_\rho - e) \leq p - \rho_0 \) where \( \rho_0 = \min_{x \in V} \rho(x) \).

(2) For a complete fuzzy graph \( K_\rho, \gamma_{cnged}(K_\rho - e) = p - \rho(x) \), where \( \rho(x) \) is obtained from \( \phi(e) = \rho(x) \land \rho(y) = \rho(y) \).

(3) For a path fuzzy graph \( P_\rho, \gamma_{cnged}(P_\rho) = \gamma_{ged}(P_\rho) \leq \frac{p}{3} + 1 \).

**Theorem 4.1.** Let \( G(\rho, \phi) \) be a fuzzy graph with pendent vertex, then \( \gamma_{cnged}(G) = \gamma_{ged}(G) \) or \( \gamma_{ged}(G) + 1 \).

**Proof.** Let \( G(\rho, \phi) \) be a fuzzy graph and \( D \) be a \( \gamma_{ged} \)-set. Let \( x \) be a pendent vertex in \( G \). If \( x \) and its support vertex \( y \) is in \( D \), at that point \( V - D \) is not a g-eccentric dominating set. Subsequently, \( \gamma_{cnged}(G) = \gamma_{ged}(G) \). Assume \( x \) or its support vertex \( y \) is in \( D \), at that point \( D_1 = D \cup \{y\} \) or \( D_1 = D \cup \{x\} \) may be a g-eccentric dominating set and \( V - D_1 \) is not a g-eccentric dominating set. Consequently \( \gamma_{cnged}(G) = \gamma_{ged}(G) + 1 \). \( \square \)

**Theorem 4.2.** In the event that \( G(\rho, \phi) \) be a fuzzy graph with \( d_g(G) = 2 \), at that point \( \gamma_{cnged}(G) \leq \delta_s(G) + 1 \).
**Proof.** Let $G(\rho, \phi)$ be a fuzzy graph with $d_g(G) = 2$. At that point we have $x \in V(G)$ be such that $d_s(x) = \delta_s(G)$. Presently, let us take $D = \{x\} \cup N_s(x) = N_s[x]$. In this manner, each vertex in $V - D$ is strong neighbors to a few vertices of $N_s(x)$ and are g-eccentric to $x$. Subsequently $D$ is g-eccentric dominating set and $V - D$ is not g-eccentric dominating set, since $x$ can not dominated by any vertex in $V - D$. Hence, $\gamma_{cnged}(G) \leq \delta_s(G) + 1$. □

**Theorem 4.3.** $W_\rho$ be a fuzzy wheel graph, at that point $\gamma_{cnged}(W_\rho) \leq 4$, $|\rho^*| = n \geq 5$.

**Proof.** Let $W_\rho, |\rho^*| = n, n \geq 5$ be a fuzzy wheel graph. Let $D = \{x, y, z, w\}$, where $w$ may be a central vertex, $z$ may be a enclave vertex and $x, y$ are any two pheriperal vertices which are no strong neighbors. $D$ is a least g-eccentric dominating set. Hence, $V - D$ is not a g-eccentric dominating set. Subsequently, $D$ is a CNGED-set. Hence, $\gamma_{cnged}(W_\rho) \leq 4$. □

**Theorem 4.4.** Let $T$ be a fuzzy tree graph $G(\rho, \phi)$ such that each support vertex is strong neighbor of at least one pendent vertex. Then $\gamma_{cnged}(T) \leq U + 2$, where $U$ is the number of support vertices.

**Proof.** Let $U$ be the set of all support vertices of $G(\rho, \phi)$. Here, all the non end vertices frame a dominating set. In this manner, to create a g-eccentric dominating set we have to include at most two pendent vertices. Subsequently, the CNGED-set contains all the non conclusion vertices and at most two pendent vertices. Therefore, $V - D$ is not a g-eccentric dominating set. Hence, $\gamma_{cnged}(T) \leq U + 2$. □

**Theorem 4.5.** Let $K_\rho$ be a completel fuzzy graph, $|\rho^*| = n, n$ is even. Let $G(\rho, \phi)$ be a fuzzy graph obtained from the complete fuzzy graph $K_\rho$ by deleting edges of linear factor. Then $\gamma_{cnged}(G) \leq \frac{n}{2} + 1$.

**Proof.** Let $G(\rho, \phi)$ be a fuzzy graph obtained from a non-trivial complete fuzzy graph. let $V = \{x_1, x_2, \ldots, x_n\}$ be the vertices of $G(\rho, \phi)$ and $G(\rho, \phi) = K_\rho - \{x_1x_2, x_3x_4, \ldots, x_{n-1}x_n\}$. Then $D = \{x_1, x_3, \ldots, x_{n-1}\}$ and $V - D = \{x_2, x_4, \ldots, x_n\}$ are g-eccentric dominating sets and we know that $\gamma_{ged}(G) \leq \frac{n}{2}$. Hence, when we include one more vertex in $D$, then $D$ is CNGED-set and $V - D$ is not g-eccentric dominating set. Thus, $\gamma_{cnged}(G) \leq \frac{n}{2} + 1$. □
5. CONCLUSION

In this paper, we examined the new domination parameter complementary nil g-eccentric domination number for a few standard fuzzy graphs and theorems related to bounds for CNGED-set for fuzzy graph.

REFERENCES