ON DUAL MULTIPLIERS IN CI-ALGEBRAS

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ABSTRACT. The Concept of BCK/BCI-algebras was first introduced by Y. Imai and K. Iseki [2, 3] in 1966. These BCK/BCI algebras can be generalized into several different categories of algebras like BCH [1], BH [4], d [9], etc. Later on, dual BCK algebras [5] was introduced which paved the way for development of BE-algebras [6]. In 2010, B. L. Meng [8] introduced the idea of CI-algebras as a generalization of BE-algebras which is considered to be an important algebraic structure till date. The concept of Cartesian product has been developed in 2013 [10] which plays a key role in the development of this CI-algebras. A new concept of Absorptive CI-algebra has been developed in 2016. The idea of Multipliers in BE-algebras [7] has been utilized to develop the idea of Multipliers in CI-algebras [12] in 2019. In this paper we present, definition of Dual Multipliers in CI-algebras and talk about few examples, characteristics of this map.

1. INTRODUCTION

The Concept of BCK/BCI-algebras was first introduced by Y. Imai and K. Iseki [2, 3] in 1966. These BCK/BCI algebras can be generalized into several different categories of algebras like BCH [1], BH [4], d [9], etc. Later on, dual BCK algebras [5] was introduced which paved the way for development of BE-algebras [6]. In 2010, B. L. Meng [8] introduced the idea of CI-algebras as a generalization of BE-algebras which is considered to be an important algebraic

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structure till date. The concept of Cartesian product has been developed in 2013 [10] which plays a key role in the development of this CI-algebras. A new concept of Absorptive CI-algebra has been developed in 2016. The idea of Multipliers in BE-algebras [7] has been utilized to develop the idea of Multipliers in CI-algebras [12] in 2019. In this paper we present, definition of Dual Multipliers in CI-algebras and talk about few examples, characteristics of this map.

2. Preliminaries

**Definition 2.1.** [6]: A non-empty set $B$, equipped with a binary operation $*$ and a fixed element $1$ is said to be a BE-algebra if it satisfies the following postulates:

- $(B1) \ t \ast t = 1$,
- $(B2) \ t \ast 1 = 1$,
- $(B3) \ 1 \ast t = t$,
- $(B4) \ t \ast (u \ast v) = u \ast (t \ast v)$ for all $t, u, v \in B$.

**Definition 2.2.** [8]: A non-empty set $C$, equipped with a binary operation $*$ and a fixed element $1$ is said to be a CI-algebra if it satisfies the following postulates:

- $(C1) \ t \ast t = 1$,
- $(C2) \ 1 \ast t = t$,
- $(C3) \ t \ast (u \ast v) = u \ast (t \ast v)$ for all $t, u, v \in C$.

**Example 1.** Let $H$ be a Hilbert space and let $B(H)$ be the class of all bounded linear operators defined on $H$. Let $C \subset B(H)$ be the set of all positive invertible and commutative operators. We define a binary operation $*$ on $C$ as

$$P \ast Q =QP^{-1}, \text{ for all } P, Q \in C.$$  

Let $I$ be the identity operator on $H$. Then $I \in C$. Also for $P, Q, R \in C$, we have

- $(E1) \ P \ast P = PP^{-1} = I$
- $(E2) \ I \ast P = PI^{-1} = P$
- $(E3) \ P \ast (Q \ast R) = P \ast (RQ^{-1}) = (RQ^{-1})P - 1 = R(Q^{-1}P^{-1}) = R(P^{-1}Q^{-1})$
- $= (RP^{-1})Q^{-1} = Q \ast (RP^{-1}) = Q \ast (PR)$

This means that $(C; \ast, I)$ is a CI-algebra.

A binary relation $\leq$ in $C$ can be defined by $t \leq u$ iff $t \ast u = 1$.

**Definition 2.3.** [8]: A non-empty subset $A$ of a CI-algebra $C$ is said to be a subalgebra of $C$ if $t \in A, u \in A$ imply $t \ast u \in A$. 

Theorem 2.1. [11] Let \((C; *, 1)\) be a CI-algebra and let, the collection of all functions \(h : C \to C\) be denoted by \(G(C)\). We define a binary operation \(o\) in \(G(C)\) such that for \(h, k \in G(C)\) and \(t \in C\),

\[
(h o k)(t) = h(t) \ast k(t).
\]

where \(1\) is defined as \(1(t) = 1\) for each element \(t \in C\). Then \((G(C); o, 1')\) is a CI-algebra. Here two functions \(h, k \in G(C)\) are equal iff \(h(t) = k(t)\), for each element \(t \in C\).

Definition 2.4. Let \(h, k \in G(C)\). Then composite of \(h\) and \(k\), denoted as \(h \bullet k\), is defined as

\[
(h \bullet k)(t) = h(k(t)).
\]

Definition 2.5. A multiplier \(h \in G(C)\) is a mapping such that \(h(t \ast u) = t \ast h(u)\) for all \(t, u \in C\).

3. Dual Multipliers In CI-Algebras

Definition 3.1. A dual multiplier \(h \in G(C)\) is a mapping such that \(h(t \ast u) = h(t) \ast u\) for all \(t, u \in C\).

Note: The identity map \(I(t) = t\) is a multiplier as well as a dual multiplier.

Proposition 3.1. Suppose \(h\) is a dual multiplier defined on CI-algebra \((C; *, 1)\).

(a) If \(h(1) = e\) then \(h(t) = e \ast t\) for any \(t \in C\),

(b) If \(h(1) = 1\) then \(h\) is the identity map.

Proof.

(a) Let \(h(1) = e\). Since \(1 \ast t = t\) for any \(t \in C\), and \(h\) is a dual multiplier,

\[
h(1 \ast t) = h(t) \Rightarrow h(1) \ast t = h(t) \Rightarrow e \ast t = h(t),
\]

(b) Putting \(e = 1\) in (a), we get

\[
h(t) = 1 \ast t = t
\]

for \(t \in C\). So \(h\) is the identity map. \(\Box\)

Theorem 3.1. Composite of two dual multiplier maps is a dual multiplier.
Proof. Suppose \( h \) and \( k \) are two dual multiplier maps defined on a CI-algebra \((C; *, 1)\). Let \( t, u \in C \). Then
\[
(h \bullet k)(t * u) = h(k(t * u)) = h(k(t)) * u = (h \bullet k)(t) * u.
\]
So \( h \bullet k \) is a dual multiplier map. \( \square \)

As above we can also prove

Corollary 3.1.

(a) If \( h \) is multiplier and \( k \) is dual multiplier, then
\[
(h \bullet k)(t * u) = k(t) * h(u)
\]
(b) If \( h \) is dual multiplier and \( k \) is multiplier, then
\[
(h \bullet k)(t * u) = h(t) * k(u).
\]

Notation: For \( h \in G(C) \), let \( B_h = \{ t \in C : h(t) = t \} \).

Proposition 3.2. If \( h \) is dual multiplier then \( h(1) \neq 1 \) iff \( B_h \) is empty.

Proof. Suppose \( h \) is a dual multiplier and \( h(1) \neq 1 \). If possible, suppose \( B_h \) is non-empty and \( t \in B_h \). Now, we have \( t * t = 1 \) and so
\[
h(1) = h(t * t) = h(t) * t = t * t = 1,
\]
which contradicts our assumption that \( h(1) \neq 1 \). So \( B_h = \phi \).

Again, let us assume \( B_h \) is empty. Suppose \( h(1) = 1 \). Then \( 1 \in B_h \) which is a contradiction to the fact that \( B_h \) is empty. Hence \( h(1) \neq 1 \). \( \square \)

Proposition 3.3. If \( h \) is a dual multiplier and \( B_h \) is non-empty then \( B_h \) is a sub-algebra.

Proof. Suppose \( h \) is a dual multiplier and let \( m, n \in B_h \). We have \( h(m) = m \) and \( f(n) = n \). Now \( h(m * n) = h(m) * n = m * n \Rightarrow m * n \in B_h \).

Therefore, \( B_h \) is a sub-algebra. \( \square \)

Definition 3.2. Suppose \((C; *, 1)\) is a CI-algebra. We define an addition ‘\(+\)’ in \( C \) as
\[
t + u = (t * u) * u \text{ for all } t, u \in C.
\]

Theorem 3.2. Suppose \( h \) is a dual multiplier on a CI-algebra \( C \). Then

(i) \( B_h \) is closed w. r. t. operation ‘\(+\)’;
(ii) \( t \in B_h \) and \( t \leq u \Rightarrow u \in B_h \).
Proof. 
(i) Let \( t, u \in B_h \). Then \( h(t) = t \) and \( h(u) = u \). Now 
\[
\begin{align*}
    h(t + u) &= h((t * u) * u) \\
    &= (h(t * u)) * u \\
    &= (h(t) * u) * u \\
    &= (t * u) * u \\
    &= t + u.
\end{align*}
\]
This implies that \( t + u \in B_h \) and proves the result. 
(ii) Given \( t \in B_h \) and \( t \leq u \Rightarrow h(t) = t \) and \( t * u = 1 \). Now 
\[
\begin{align*}
    h(u) &= h(1 * u) = h((t * u) * u) \\
    &= (h(t * u)) * u \\
    &= (h(t) * u) * u \\
    &= (t * u) * u \\
    &= 1 * u = u.
\end{align*}
\]
This proves that \( u \in B_h \). \( \square \)

References


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