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RESULTS ON RESOLVABILITY AND METRIC DIMENSION IN GRAPHS

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ABSTRACT. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$ and a vertex $v \in V$, the metric representation of v with respect to W is a k-vector, which is defined as $r(v/W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(u, v)$ represents the distance between the vertices u and v. The set W is called a resolving set for G if $r(u/W) = r(v/W)$ implies that $u = v$ for all $u, v \in V(G)$. In this paper, the total number of resolving sets for a path graph P_n is obtained. Established that total number of resolving sets in a simple connected graph G is greater than or equal to $2^{n-k} - 1$, where $k = \dim(G)$. Discussed about $K_{m,n}$ $m \geq 2, n > m - 2$, does not admit independent basis. Established that every basis for hypercube Q_3 is either independent or connected and every basis of Petersen graph (P) is independent. Characterized the graph G with $\dim(G) = 2$, which does not admit an independent basis.

1. INTRODUCTION

The concepts of metric dimension of a graph and its related properties such as basis were introduced by P.J. Slater [3] and independently by Harary and Melter [4]. The metric dimension of a graph is the least number of vertices in a set with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex. Slater introduced this concept by motivated from the robot navigation problem. The motivation behind this work is due to the

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large range of application of resolving sets in various fields such as navigation in robotic networks, problems of pattern recognition and image processing.

2. PRELIMINARIES

All the graphs considered in this paper are undirected, simple, finite and connected. The order and size of G are denoted by n and r respectively. We use standard terminology, the terms not defined here may found in [4], [1] and [5]. A *graph* $G = (V, E)$ consists of two sets: V, the (non empty) vertex set of the graph and E, the edge set of the graph, such that each edge e in E is assigned an unordered pair of vertices (u, v), called the end vertices of e. A graph $G = (V, E)$ is said to be a graph on n vertices and r edges if $|V| = n$ and $|E| = r$ and are respectively known as order and size of G. A graph with finite number of vertices is called finite graph. A graph with no loops and multiple edges is called a *simple graph*. A graph $G = (V, E)$ is *connected* if there is a path joining each pair of nodes. A *component* of a graph is a maximal connected subgraph. If a graph has only one component it is connected, otherwise it is *disconnected*. For a connected graph $G = (V, E)$, the *distance* between any two vertices a and b, denoted by $d(a, b)$, is the length of the shortest path joining a and b. Let $G = (V, E)$ be a connected, undirected graph and $v_1, v_2, v_3 \in V$. A vertex v is said to resolve the vertices v_1 and v_3 if the distance of v_1 from v_2 is different from distance of v_3 from v_2. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$ and for any vertex $v \in V$, the (metric)representation of v with respect to W is the k-vector which is denoted and defined as $r(v/W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is called a resolving set for G if $r(v_1|W) = r(v_2|W)$ implies that $v_1 = v_2$ for all $v_1, v_2 \in V(G)$. A resolving set of minimum cardinality for a graph G is called a minimum resolving set. A minimum resolving set is usually called a basis for G. The minimum cardinality of a resolving set of G is called the metric dimension of G and is denoted by $\dim(G)$. A resolving set W is called a minimal resolving set of a connected graph G if no proper subset of W is a resolving set of G. A basis W for a graph G is said to connected if the subgraph induced by W is connected. A basis W of G is said to be independent if no two vertices of W are adjacent. [2] A connected graph G of order $n > 2$ has dimension $n - 1$ if and only if $G = K_n$. [5] For Q_n,

$$\dim(Q_n) \text{ is } n, \text{if } n \leq 4;\ n-1, \text{if } 5 \leq n \leq 6;\ 6, \text{if } 7 \leq n \leq 8;\ 7, \text{if } 9 \leq n \leq 10.$$

[3] Let G be a connected graph of order $n \leq 4$. Then $\dim(G) = n - 2$ if and only if

$G = K_{r,s}(r, s \geq 1), G = K_r + \bar{K}_s(r \geq 1, s \geq 2),$ or $G = K_r + (K_1 \cup K_s)(r, s \geq 1)$.

3. Main Results

Theorem 3.1. *For Paths $P_n, n \geq 2$, total number of resolving sets is $2 + \sum_{k=2}^{n-1} nC_k$.*

Proof. For $n \geq 2$, $\dim(P_n) = 1$ and P_n has only 2 bases, that are the end vertices. Since every two element subsets of $V(P_n)$ is a resolving set, the number of resolving sets of order 2 of P_n is nC_2. Similarly number of resolving sets of order 3 of P_n is nC_3, number of resolving sets of order 4 of P_n is nC_4 and so on. Hence the total number of resolving sets of P_n is $2 + nC_2 + nC_3 + \ldots + nC_{n-1}$
$= 2 + \sum_{k=2}^{n-1} nC_k$. □

Theorem 3.2. *Total number of resolving sets in a simple connected graph G is greater than or equal to $2^{n-k} - 1$, where $k = \dim(G)$.*

Proof. Let G be a simple connected graph with $|V| = n$, $\dim(G) = k$ and W be a resolving set with $|W| = k$. Let $W_1 \supset W$ and $W_1 \neq V$.

Claim: W_1 is a resolving set. Since $W \subset W_1$, the metric representation of every vertex of G with respect to W_1, the first k-positions have distinct coordinates and the remaining position may or may not be equal. So all the vertices get different metric representation with respect to W_1, therefore W_1 is a resolving set of G. Hence the claim. Since $G - \langle W \rangle$ has $n - k$ vertices, total number of subset with respect to $G - \langle W \rangle$ is 2^{n-k}. So there are $2^{n-k} - 1$ supersets of W as resolving set. Therefore total number of resolving sets of a graph G with $\dim(G)$ is greater than or equal to $2^{n-k} - 1$. □

Theorem 3.3. *$K_{m,n}, m \geq 2, n > m - 2$, does not admit independent basis.*

Proof. The set of vertices of $K_{m,n}$ can be taken as $V = V_1 \cup V_2$, where $|V_1| = m$ and $|V_2| = n$, $V_1 \cap V_2 = \emptyset$. Since $\dim(K_{m,n}) = m + n - 2$ and the possible independent basis consists of $m - 1$ vertices from V_1 and any $n - 1$ vertices from V_2, which is not independent. Therefore $K_{m,n}, m \geq 2, n > m - 2$ does not admit an independent basis. □

Theorem 3.4. *Every basis for hypercube Q_3 is either independent or connected.*

Proof. Note that $\dim(Q_3) = 3$. There are three types of subsets W of $V(Q_3)$ such that $|W| = 3$. The subgraph induced by W is any of P_3, $\overline{K}_3$ and $P_2 \cup P_1$.
Case 1: If $\langle W \rangle = P_3$, then W is a basis and also connected basis. Note that the vertex set of each subgraph W of Q_3 of the form P_3 is a connected basis.
Case 2: If $\langle W \rangle = \overline{K}_3$,then W is a basis and also an independent basis. Note that the vertex set of each subgraph W of Q_3 of the form $\overline{K}_3$ is an independent basis.
Case 3: If $\langle W \rangle = P_2 \cup P_1$, then W is not a basis. Since if possible, assume W is a basis, then atleast two pair of vertices with have same metric representation. For example in Figure 1, let $W = \{v_1, v_2, v_8\}$ then the vertices v_3 and v_6 have the same metric representation $(2, 1, 2)$ and the vertices v_4 and v_5 have the same metric representation $(1, 2, 1)$. Therefore W is not a basis. In general any basis of Q_3 is either connected or independent. □

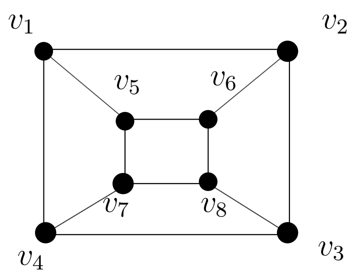

FIGURE 1. Hypercube Q_3.

Theorem 3.5. *Every basis of Petersen graph* (P) *is independent.*

Proof. For Petersen graph, $\dim(P) = 3$ and $d(u, v) \leq 2 \; \forall u, v \in V(P)$.
Claim: For $W = \{v_1, v_2, v_3\} \subset V(P)$ and if $\langle W \rangle$ is either P_3 or $P_2 \cup P_1$, then W is not a basis for P. *Proof for the Claim*: Consider $\langle W \rangle$ is either P_3 or $P_2 \cup P_1$.
Case 1: If $\langle W \rangle = P_3$. Without loss of generality we assume that if P_3 is a path with internal vertex as v_1, then three such paths exist, say $v_5 - v_1 - v_6$, $v_2 - v_1 - v_6$ and $v_5 - v_1 - v_2$. If $W = \{v_1, v_5, v_6\}$, then the vertices v_3 and v_7 have the same metric representation $(2, 2, 2)$ with respect to W. If $W = \{v_1, v_2, v_6\}$, then the vertices v_4 and v_{10} have the same metric representation $(2, 2, 2)$ with respect to W. If $W = \{v_1, v_2, v_5\}$, then the vertices v_4 and v_{10} have the same metric representation $(2, 2, 1)$ with respect to W. So in general if P_3 is a path with internal vertex v_i, then $\langle W \rangle = P_3$, and W is not a resolving set of P.

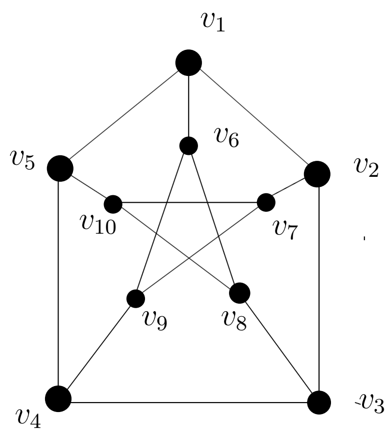

FIGURE 2. Petersen graph (P).

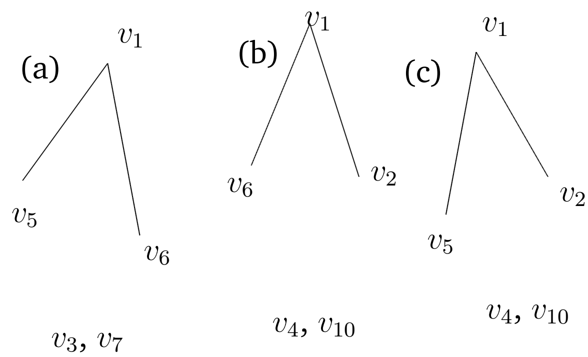

FIGURE 3. Sub graphs of P.

Case 2: If $\langle W \rangle = P_2 \cup P_1$ .Let P_2 be the path: $v_1 - v_2$. Now the vertex of P_1 can be any one from the set $\{v_4, v_8, v_9, v_{10}\}$, which is the set of vertices not adjacent to v_1 and v_2. Therefore the following possibilities arises.

If $W = \{v_1, v_2, v_4\}$, then the vertices v_7 and v_{10} have the same metric representation $(2, 2, 2)$. If $W = \{v_1, v_2, v_8\}$, then the vertices v_4 and v_9 have the same metric representation $(2, 2, 2)$. If $W = \{v_1, v_2, v_9\}$, then the vertices v_8 and v_{10} have the same metric representation $(2, 2, 2)$. If $W = \{v_1, v_2, v_{10}\}$, then the vertices v_4 and v_9 have the same metric representation $(2, 2, 2)$. In general consider if $\langle W \rangle = P_2 \cup P_1$ with $P_2 = v_i - v_j$, then W is not a resolving set. $\square$

Theorem 3.6. *A graph G with $\dim(G) = 2$ does not admit an independent basis if and only if G is either K_3, C_4 or $P_1 + P_3$.*

Proof. Assume that G is a simple connected graph with n vertices, $\dim(G) = 2$ and it does not have an independent basis. Therefore there are two possibilities.
Case (1) $V(G)$ does not have an independent subset of vertices. In this case every vertex of G is adjacent to each other. So it must be a complete graph K_n of order n and hence $\dim(G) = n - 1 \Rightarrow 2 = n - 1$ (by assumption). $\Rightarrow n = 3 \Rightarrow G = K_3$.
Case (2). There exists an independent two element subset of $V(G)$ which is not a resolving set. If $W = \{w_1, w_2\}$ be an independent subset of V and is not a basis, then there exist at least two vertices u_1 and v_1 in $V - W$ such that $d(u_1, W) = d(v_1, W)$.
Claim: (a) $d(u_1, w_1) = d(u_1, w_2) = d(v_1, w_1) = d(v_1, w_2) = 1$. (b) There does not exist more than one such pair of vertices. Proof of the Claim: (a) If possible assume that $d(u_1, w_1) = d(u_1, w_2) = d(v_1, w_1) = d(v_1, w_2) = k > 1$. So the graph is as shown in the Figure 4(B). Label the vertices between w_1 and u_1 as $u_{11}, u_{12}, u_{13}, \ldots, u_{1k}$, the vertices between u_1 and w_2 are labeled as $u_{21}, u_{22}, u_{23}, \ldots, u_{2k}$, the vertices between w_1 and v_1 as $v_{11}, v_{12}, v_{13}, \ldots, v_{1k}$ and the vertices between v_1 and w_2 as $v_{21}, v_{22}, v_{23}, \ldots, v_{2k}$. Then $W = \{w_1, u_1\}$ form an independent basis, since with respect to W the vertices in $V - W$ get the label as $(1, k-1), (2, k-2), \ldots, (k-1, 1), (k+1, 1), (k+2, 2), \ldots, (2k-1, k-1), (2k-2, k-2), \ldots, (1, k+1)$. Which is not possible by assumption. So the graph is as shown either in Figure 4(A) or in Figure 4(C) or in Figure 4(E) but not as in Figure 4(D). Therefore $d(u_1, w_1) = d(u_1, w_2) = d(v_1, w_1) = d(v_1, w_2) = 1$.
Claim: (b) If possible assume there exist k pair of vertices (u_i, v_i) such that $d(u_i, W) = d(v_i, W)$, $u_i, v_i \in V - W$. Then the graph is as shown in Figure 4(C). So we can find at least one independent set $W \subset V$ such that $|W| = 2$, say $N = \{u_i, v_j\}, i, j = 1, 2, \ldots, k$, form an independent basis. Which is not possible by assumption. So the graph must be as in Figure 4(A) or in Figure 4(E). So it is C_4 or $P_1 + P_3$, hence the proof. Converse part is trivial, since $\dim(K_3) = 2$, $\dim(C_4) = 2$ and $\dim(P_1 + P_3) = 2$ and no basis of each of these graphs is independent. $\square$

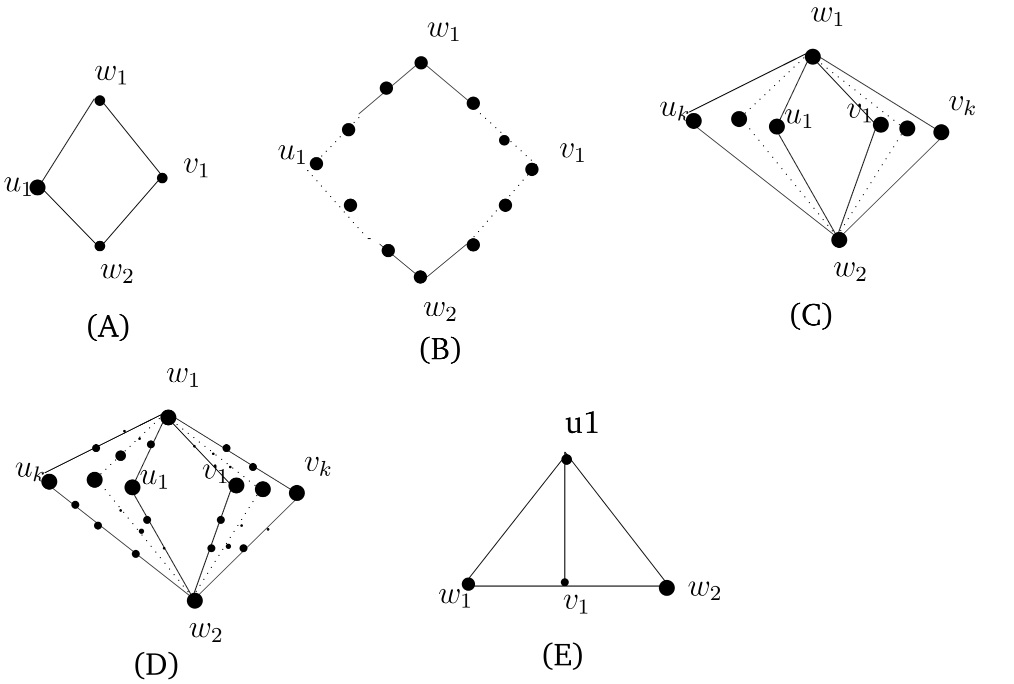

FIGURE 4. Graphs with $\dim(G) = 2$ and do not possess independent basis.

4. CONCLUSION

In this paper determined total number of resolving sets for some classes of graphs. Determined total number of basis in a graph and studied about those graphs which do not possess independent basis. Characterized graph with

$$\dim(G) = 2,$$

which do not admit independent basis.

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