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A COMMON FIXED POINT THEOREM USING COMPATIBLE MAPS OF TYPE (γ) AND (δ)

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ABSTRACT. In this article, we prove a common fixed point theorems using compatible mapping of type (γ) and (δ) in fuzzy metric spaces.

1. INTRODUCTION

The generalization of the commuting mapping concept is compatible mapping which is introduced by Gerald Jungck [3]. This concept was generalized to fuzzy metric spaces by Mishra et al. [8]. Y. J. Cho introduced the concept of compatible mapping of type (α) [1] and compatible mapping of type (β) [2]. The authors defined intuitionistic (ψ, η) contractive mapping in [7]. Using the definition of ψ, we gave a common fixed point theorem. Also, The authors introduced compatible mapping of type (γ) and compatible mapping of type (δ) in [6]. Further, the theorem is discussed for two different types of compatible mappings. In this paper [7], ψ is defined as follows.

Definition 1.1. *Let Ψ be the class of all mappings $\psi : [0,1] \to [0,1]$ such that:*

(i) ψ *is non-decreasing and* $\lim_{n\to\infty} \psi^n(s) = 1, \forall s \in (0,1]$;
(ii) $\psi(s) > s, \forall s \in (0,1)$;
(iii) $\psi(1) = 1$.

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Also in [6], compatible mapping of type (γ) and compatible mapping of type (δ) are defined as follows:

Definition 1.2. *Let $(U, \mu, *)$ be a fuzzy metric space. We say that the two self mappings A and B are called:*

(a) compatible of type (γ) if for all $t > 0, \lim_{n\to\infty} \mu(AAu_n, Bw, t) = 1$ and $\lim_{n\to\infty} \mu(BBu_n, Aw, t) = 1$ whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some $w \in U$.

(b) compatible of type (δ) if for all $t > 0, \lim_{n\to\infty} AAu_n = \lim_{n\to\infty} ABu_n = Bw$ and $\lim_{n\to\infty} BBu_n = \lim_{n\to\infty} BAu_n = Aw$, whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some $w \in U$.

2. Preliminaries

Definition 2.1. *[5] Let U be a nonempty set and $*$ a continuous t-norm. A fuzzy set μ on $U^2 \times [0, \infty)$ is called a fuzzy metric on U if for all $u, v, w \in U$ and $s, t > 0$, the following conditions hold:*

(i) $\mu(u, v, 0) = 0$;
(ii) $\mu(u, v, t) = 1$ iff $u = v$;
(iii) $\mu(u, v, t) = \mu(v, u, t)$;
(iv) $\mu(u, w, t + s) \geq \mu(u, v, t) * \mu(v, w, s)$;
(v) $\mu(u, v, .) : [0, \infty) \to [0, 1]$ is left continuous.

Then $(U, \mu, *)$ is said to be a fuzzy metric space.

Definition 2.2. *[4] Let $(U, \mu, *)$ be a fuzzy metric space. A sequence $\{u_n\}$ in U is called:*

(a) convergent to a point $u \in U$ iff $\lim_{n\to+\infty} \mu(u_n, u, t) = 1$ for all $t > 0$,
(b) Cauchy if $\lim_{n\to\infty} \mu(u_n, u_{n+a}, t) = 1$ for all $t > 0$ and $a > 0$.

Definition 2.3. *[4] A fuzzy metric space $(U, \mu, *)$ is called complete if every Cauchy sequence in U is convergent.*

Definition 2.4. *[8] In a fuzzy metric space $(U, \mu, *)$, two self mappings A and B are called compatible if $\lim_{n\to\infty} \mu(ABu_n, BAu_n, t) = 1$ whenever u_n is a sequence in U and if for all $t > 0$, $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some $w \in U$.*

Definition 2.5. *[9] Two self maps A and B of a fuzzy metric space $(U, \mu, *)$ are said to be reciprocally continuous on U if $\lim_{n\to\infty} ABu_n = Aw$ and $\lim_{n\to\infty} BAu_n = Bw$ whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w in U.*

Proposition 2.1. *[6] Let A and B be compatible mappings of a fuzzy metric space $(U, \mu, *)$ into itself. If $Aw = Bw$ for some $w \in U$, then $ABw = BAw$.*

Proposition 2.2. *[6] Let A and B be compatible mapping of type (δ) of a fuzzy metric space $(U, \mu, *)$ into itself. Let one of A and B be continuous. Suppose that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some $w \in U$. Then $Aw = Bw$.*

Lemma 2.1. *[8] If A and B are compatible mappings on a fuzzy metric space U and $Au_n, Bu_n \to w$ for some w in U($u_n$being a sequence in U) then $ABu_n \to Bw$ provided B is continuous (at w).*

3. Main Results

Theorem 3.1. *Let A and B be self maps on a complete fuzzy metric space U and $\psi \in \Phi$ such that satisfy the following conditions:*

- *(I)* $A(U) \subset B(U)$,
- *(II)* $\mu(Au, Av, t) \geq \psi(\mu(Bu, Bv, t))$ *for all* $u, v \in U$ *and* $t > 0$,
- *(III)* *A or B is continuous.*
- *(IV)* *the sequence u_n and v_n in U are such that $\{u_n\} \to u, \{v_n\} \to v, t > 0$ implies $\mu(u_n, v_n, t) \to \mu(u, v, t)$.*

Assume that A and B are compatible. Then A and B have a unique common fixed point in U.

Proof. Let $u_0 \in U$ and $A(U) \subset B(U)$ define a sequence u_n in U, for all $n \in N$ as follows:

$$Au_n = B(u_{n+1}).$$

Then for all $t > 0$ and suppose n is odd,

$$\begin{aligned}\mu(Au_n, Au_{n+1}, t) &\geq \psi\mu(Bu_n, Bu_{n+1}, t)\\ &= \psi\mu(Au_{n-1}, Au_n, t)\\ &\geq \psi^2(\mu(Bu_{n-1}, Bu_n, t))\\ &\dots\\ &\geq \psi^n(\mu(Au_0, Au_1, t)).\end{aligned}$$

That is,$\mu(Au_n, Au_{n+1}, t) \geq \psi^n(\mu(Au_0, Au_1, t))$. By taking limit as $n \to \infty$, and since $\lim_{n\to\infty} \psi^n(s) = 1$, for all $s \in (0, 1]$,

$$\lim_{n\to\infty} \mu(Au_n, Au_{n+1}, t) = 1.$$

For all $a > 0$,

$$\mu(Au_n, Au_{n+a}, t) \geq \mu(Au_n, Au_{n+1}, t/a) * \dots * \mu(Au_{n+a-1}, Au_{n+a}, t/a).$$

By taking limit $n \to \infty$,

$$\begin{aligned}\lim_{n\to\infty} \mu(Au_n, Au_{n+a}, t) &\geq \lim_{n\to\infty} \mu(Au_n, Au_{n+1}, t/a) * \dots * \lim_{n\to\infty} \mu(Au_{n+a-1}, Au_{n+a}, t/a)\\ &\geq 1 * \dots * 1\\ &= 1.\end{aligned}$$

That is,

$$\lim_{n\to\infty} \mu(Au_n, Au_{n+a}, t) = 1.$$

Similarly suppose n is even, $\mu(Au_n, Au_{n+1}, t) \geq \psi^n(\mu(Bu_0, Bu_1, t))$. By taking limit as $n \to \infty$, and since $\lim_{n\to\infty} \psi^n(s) = 1$, for all $s \in (0, 1]$,

$$\lim_{n\to\infty} \mu(Au_n, Au_{n+1}, t) = 1.$$

Also, we can prove

$$\lim_{n\to\infty} \mu(Au_n, Au_{n+a}, t) = 1.$$

Hence,$\{Au_n\}$ is a Cauchy sequence in U.

Since $(U, \mu, *)$ is a complete fuzzy metric space, there exists $w \in U$ such that $\lim_{n\to\infty} \mu(Au_n, w, t) = 1$ and $\lim_{n\to\infty} \mu(Bu_n, w, t) = 1$, for each $t > 0$. Suppose A is continuous, since A and B are compatible and by Lemma 2.1, $BAu_n \to Aw$.

Now,

$$\mu(Au_n, AAu_n, t) \geq \psi(\mu(Bu_n, BAu_n, t)).$$

By taking limit as $n \to \infty$,

$$\mu(w, Aw, t) \geq \psi(\mu(w, Aw, t)) \geq \mu(w, Aw, t).$$

This is possible only when $\mu(w, Aw, t) = 1$. That is $Aw = w$. Since $A(U) \subset B(U)$ there exists w_1 in U such that $w = Aw = Bw_1$. From

$$\mu(AAu_n, Aw_1, t) \geq \psi(\mu(BAu_n, Bw_1, t)),$$

by taking limit as $n \to \infty$,

$$\mu(Aw, Aw_1, t) \geq \psi(\mu(Aw, Bw_1, t)) = \psi(1) = 1.$$

That is $Aw_1 = Bw_1$.

Now, we have $Aw = Aw_1$. By Proposition 2.1, $ABw_1 = BAw_1$.

$$\mu(Aw, Bw, t) = \mu(ABw_1, BAw_1, t) = 1.$$

Hence, $Aw = Bw = w$. Hence A and B have a common fixed point in U.

Uniqueness:

Assume $\overline{w} \neq w$ for some $\overline{w} \in U$, is another common fixed point in U. Then for $t > 0$, we have,

$$\begin{aligned} \mu(w, \overline{w}, t) &= \mu(A(w), A(\overline{w}), t) \\ &\geq \psi(\mu(B(w), B(\overline{w}), t)) \\ &\cdots \\ &\geq \psi^n(\mu(B(w), B(\overline{w}), t)). \end{aligned}$$

Taking limit as $n \to \infty$ and by our assumption,

$$\mu(w, \overline{w}, t) \geq \lim_{n \to \infty} \psi^n(\mu(B(w), B(\overline{w}), t)) = 1.$$

That is, $\mu(w, \overline{w}, t) = 1$. Therefore, $w = \overline{w}$. Hence A and B have a unique common fixed point in U. □

Example 1. *Let $U = [0, \infty)$ with the metric d defined by $d(u, v) = |u - v|$, define $\mu(u, v, t) = \frac{t}{t + d(u,v)}$, for all $u, v \in U$ and $t > 0$. Note that, $(U, \mu, *)$ where $a * b = ab$ is a complete fuzzy metric space.*

The maps $A, B : U \to U$ are defined by $A(u) = \frac{2+u}{3}$ and $B(u) = u$. Let $u_n = (1 - \frac{1}{n})$. Then

$$\begin{aligned}\lim_{n\to\infty} \mu(ABu_n, BAu_n, t) &= \lim_{n\to\infty} \mu\left(Au_n, B\frac{2+u_n}{3}, t\right)\\ &= \lim_{n\to\infty} \mu\left(\frac{2+u_n}{3}, \frac{2+u_n}{3}, t\right)\\ &= 1,\end{aligned}$$

i.e., $\lim_{n\to\infty} \mu(ABu_n, BAu_n, t) = 1$,

$$\lim_{n\to\infty} Au_n = \lim_{n\to\infty} \frac{2+u_n}{3} = \lim_{n\to\infty} \frac{2+(1-\frac{1}{n})}{3} = 1$$

and

$$\lim_{n\to\infty} Bu_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

Therefore, A and B are compatible mapping. Also $AU \subset BU$ and B is continuous.

Now, define the map $\psi : [0,1] \to [0,1]$ by $\psi(s) = \frac{2s}{s+1}$ for each $s \in [0,1]$ and $\psi \in \Phi$. Then

$$\mu(A(u), A(v), t) \geq \psi(\mu(B(u), B(v), t))$$

if

$$\mu\left(\frac{2+u}{3}, \frac{2+v}{3}, t\right) \geq \psi(\mu(u, v, t)),$$

or equivalently if

$$\begin{aligned}\frac{t}{t + d(\frac{2+u}{3}, \frac{8-v}{3})} &\geq \frac{\frac{2t}{t+d(u,v)}}{\frac{t}{t+d(u,v)} + 1}\\ \Leftarrow \frac{t}{t + \left|\frac{2+u}{3} - \frac{2+v}{3}\right|} &\geq \frac{\frac{2t}{t+|u-v|}}{\frac{t}{t+|u-v|} + 1}\\ \Leftarrow \frac{t}{t + \frac{|u-v|}{3}} &\geq \frac{t}{t + \frac{|u-v|}{2}}\\ \Leftarrow t + \frac{|u-v|}{2} &\geq t + \frac{|u-v|}{3}\\ \Leftarrow 3 &\geq 2.\end{aligned}$$

All the conditions of the previous theorem are verified. Then, 1 is the unique fixed point. Hence, A and B have the unique common fixed point in U.

Now, we prove the following theorem for compatible of type (γ).

Theorem 3.2. *Let A and B be self maps on a complete fuzzy metric space U and $\psi \in \Phi$ such that satisfy the above conditions (I), (II) and (IV). Assume that A and B are reciprocally continuous and compatible of type (γ). Then A and B have a unique common fixed point in U.*

Proof. From the previous theorem, $\{Au_n\}$ and $\{Bu_n\}$ are a Cauchy sequences in U. Since $(U, \mu, *)$ is a complete fuzzy metric space, there exists $w \in U$ such that $\lim_{n\to\infty} \mu(Au_n, w, t) = 1$ and $\lim_{n\to\infty} \mu(Bu_n, w, t) = 1,$ for each $t > 0.$ Since A and B are compatible of type (γ), we have $AAu_n \to Bw$ and $BBu_n \to Aw$ as $n \to \infty$. Also since A and B are reciprocally continuous, $ABu_n \to Aw$ and $BAu_n \to Bw$ as $n \to \infty$. We claim that $Aw = Bw$. Indeed, from

$$\mu(AAu_n, ABu_n, t) \geq \psi(\mu(BAu_n, BBu_n, t))$$

by taking limit as $n \to \infty$, we receive

$$\mu(Bw, Aw, t) \geq \psi(\mu(Bw, Aw, t)) \geq \mu(Bw, Aw, t).$$

It is possible only when $\mu(Bw, Aw, t) = 1$. That is, $Aw = Bw$.

Now, from

$$\mu(Au_n, AAu_n, t) \geq \psi(\mu(Bu_n, BAu_n, t))$$

by taking limit as $n \to \infty$, we have

$$\mu(w, Bw, t) \geq \psi(\mu(w, Bw, t)) \geq (\mu(w, Bw, t)).$$

This is possible only when $\mu(w, Bw, t) = 1$. That is $Bw = w$.

Hence $Aw = Bw = w$.

Easily, we can verify the uniqueness as in the previous theorem. □

Finally, we prove the following theorem for compatible of type (δ).

Theorem 3.3. *Let A and B be self maps on a complete fuzzy metric space U and $\psi \in \Phi$ such that satisfy the above conditions (I), (II), (III) and (IV). Assume that A and B are compatible of type (δ). Then A and B have a unique common fixed point in U.*

Proof. From the Theorem 3.1, $\{Au_n\}$ and $\{Bu_n\}$ are a Cauchy sequences in U. Since $(U, \mu, *)$ is a complete fuzzy metric space, there exists $w \in U$ such that $\lim_{n\to\infty} \mu(Au_n, w, t) = 1$ and $\lim_{n\to\infty} \mu(Bu_n, w, t) = 1,$ for each $t > 0.$ Since A

and B are compatible of type (δ) and one of A and B is continuous, by Proposition 2.2, $Aw = Bw$. Now, from

$$\mu(Au_n, AAu_n, t) \geq \psi(\mu(Bu_n, BAu_n, t)),$$

by taking limit as $n \to \infty$,

$$\mu(w, Bw, t) \geq \psi(\mu(w, Aw, t))$$

Since $Aw = Bw$,

$$\mu(w, Aw, t) \geq \psi(\mu(w, Aw, t)) \geq \mu(w, Aw, t).$$

This is possible only when $\mu(w, Aw, t) = 1$. That is, $Aw = w$. Hence $Aw = Bw = w$.

Easily, we can verify the uniqueness as in the Theorem 3.1. □

Remark 3.1. *Example 1 is also suitable for Theorem 3.2 and Theorem 3.3.*

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