Abstract. A graph $G$ with $p$ vertices and $q$ edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $\{1, 2, ..., q + 1\}$ in such a way that each edge $e = uv$ is labeled with
\[
f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u) + f(v)} \right\rceil \quad \text{or} \quad \left\lfloor \frac{2f(u)f(v)}{f(u) + f(v)} \right\rfloor,
\]
then the edge labels are distinct. In this case $f$ is called Harmonic mean labeling of $G$. In this paper we prove that some families of graphs such as $H$- super subdivision of path $HSS(P_n)$, $HSS(P_n) \odot K_1$, $HSS(P_n) \odot K_2$, $HSS(P_n) \odot K_2^2$ are harmonic mean graphs.

1. Introduction

Let $G = (V, E)$ be a $(p, q)$ graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges, where $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of the graph $G$. In this paper, we consider the graphs which are simple, finite and undirected. For graph theoretic terminology and notations we refer to S.Arumugam [1].

The concept of graph labeling was introduced by Rosa in 1967. A detailed survey of graph labeling is available in Gallian [3]. The concept of Harmonic mean labeling of graph was introduced by S. Somasundaram, R. Ponraj and S.S.

\[1^{\text{corresponding author}}\]

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Sandhya and they investigated the existence of harmonic mean labeling of several family of graph. This was further studied by many authors. We have proved Harmonic mean labeling of subdivision graphs such as $P_n \odot K_1$, $P_n \odot \overline{K_2}$, H-graph, crown, $C_n \odot K_1$, $C_n \odot \overline{K_2}$, quadrilateral snake, Triangular snake and also proved Harmonic mean labeling of some graphs such as Triple triangular snake $T(T_n)$, Alternate Triple triangular snake $A[T(T_n)]$, Triple quadrilateral snake $T(Q_n)$, Alternate Triple quadrilateral snake $A[T(Q_n)]$, Twig graph $T(n)$, balloon triangular snake $T_n(C_m)$, and key graph $K_y(m, n)$.

Other valuable references are [4–7].

The following definitions are useful for the present investigation.

**Definition 1.1.** [8] A Graph $G = (V, E)$ with $p$ vertices and $q$ edges is called a Harmonic mean graph if it is possible to label the vertices $v \in V$ with distinct labels $f(v)$ from $\{1, 2, ..., q + 1\}$ in such a way that when each edge $e = uv$ is labeled with

$$f(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u) + f(v)} \right\rfloor \quad \text{or} \quad \left\lceil \frac{2f(u)f(v)}{f(u) + f(v)} \right\rceil$$

then the resulting edge labels are distinct. In this case $f$ is called Harmonic mean labeling of $G$.

**Definition 1.2.** [2] Let $G$ be a $(p, q)$ graph. A graph obtained from $G$ by replacing each edge $e_i$ by a H-graph in such a way that the ends $e_i$ are merged with a pendent vertex point in $P_2$ and a pendent vertex point in $P'_2$ is called H-Super Subdivision of $G$ and it is denoted by $HSS(G)$ where the H-graph is a tree on 6 vertices in which exactly two vertices of degree 3.

**Definition 1.3.** [2] The corona $G_1 \odot G_2$ of two graphs $G_1$ and $G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ (which has $p_1$ vertices) and $p_1$ copies of $G_2$ and then joining the $i^{th}$ point of $G_1$, to every points in the $i^{th}$ copy of $G_2$.

In this paper we prove that H- super subdivision of path $HSS(P_n)$, $HSS(P_n) \odot K_1$, $HSS(P_n) \odot \overline{K_2}$, $HSS(P_n) \odot K_2$ are harmonic mean graphs.

## 2. Harmonic mean labeling of graphs

**Theorem 2.1.** The H- super subdivision of path $HSS(P_n)$is a harmonic mean graphs.
Proof. Let $HSS(P_n)$ be the H-super subdivision of path graph whose vertex set
\[ V(G) = \{u_i, v_i, x_i, y_i / 1 \leq i \leq n\} \cup w_{n+1} \]
and the edge set
\[ E(G) = \{u_iu_i, u_iv_i, v_iy_i, w_iu_i, v_iw_i+1 / 1 \leq i \leq n\}. \]
Define a function $f : V \to \{1, 2, ..., q + 1\}$ by
\[
\begin{align*}
  f(u_i) &= 5i - 3 & \text{for } 1 \leq i \leq n \\
  f(v_i) &= 5i & \text{for } 1 \leq i \leq n \\
  f(x_1) &= 1 \\
  f(x_i) &= 5i - 2 & \text{for } 2 \leq i \leq n \\
  f(y_i) &= 5i - 1 & \text{for } 1 \leq i \leq n \\
  f(w_1) &= 3 \\
  f(w_i) &= 5i - 4 & \text{for } 2 \leq i \leq n
\end{align*}
\]
Then the resulting edge labels are distinct. Thus $f$ provides a harmonic mean labeling of graph $G$.

Hence $G$ is a harmonic mean graph. \hfill \Box

Example 1. A harmonic mean labeling of graph $G$ obtained by H- super subdivision of path $HSS(P_4)$ are given in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

Theorem 2.2. The H- super subdivision of path $HSS(P_n) \odot K_1$ is a harmonic mean graph.

Proof. Let $HSS(P_n) \odot K_1$ be the H- super subdivision of path graph whose vertex set
\[ V(G) = \{u_i, v_i, w_i, r_i, s_i, t_i, x_i, y_i, p_i, q_i / 1 \leq i \leq n\} \cup \{u_{n+1}, v_{n+1}, w_{n+1}, t_{n+1}\} \]
and the edge set

\[ E(G) = \{v_iu_{i+1}, v_iw_i, r_iu_{i+1}, v_i, t_iw_i, x_iv_i, y_iu_{i+1}, x_ip_i, y_iq_i / 1 \leq i \leq n \} \]
\[ \cup \{w_iu_i / 2 \leq i \leq n \} \cup \{u_1v_1, u_nv_n, u_bw_n, w_nt_n \}. \]

Define a labeling \( f : V(G) \to \{1, 2, ..., q + 1\} \) by

\[ f(u_1) = 1 \]
\[ f(u_i) = 10i - 11 \text{ for } 2 \leq i \leq n + 1 \]
\[ f(v_i) = 10i - 6 \text{ for } 1 \leq i \leq n \]
\[ f(v_{n+1}) = 10n + 2 \]
\[ f(x_i) = 10i - 5 \text{ for } 1 \leq i \leq n \]
\[ f(y_i) = 10i - 2 \text{ for } 1 \leq i \leq n \]
\[ f(p_i) = 10i - 4 \text{ for } 1 \leq i \leq n \]
\[ f(q_i) = 10i - 3 \text{ for } 1 \leq i \leq n \]
\[ f(w_1) = 3 \]
\[ f(w_i) = 10i - 8 \text{ for } 2 \leq i \leq n \]
\[ f(w_{n+1}) = 10n \]
\[ f(r_i) = 10i \text{ for } 1 \leq i \leq n \]
\[ f(s_i) = 10i + 3 \text{ for } 1 \leq i \leq n \]
\[ f(t_1) = 2 \]
\[ f(t_i) = 10i - 9 \text{ for } 2 \leq i \leq n \]

Then the resulting edge labels are distinct. Thus \( f \) provides a harmonic mean labeling of graph \( G \).

Hence \( G \) is a harmonic mean graph. \( \square \)

**Example 2.** A harmonic mean labeling of graph \( G \) obtained by \( H \)-super subdivision of path \( HSS(P_5) \odot K_1 \) are given in Figure 2.

**Theorem 2.3.** The \( H \)-super subdivision of path \( HSS(P_n) \odot K_2 \) is a harmonic mean graph.
Proof. Let $HSS(P_n) \odot K_2$ be the H- super subdivision of path graph whose vertex set
\[ V(G) = \{u_i, v_i, w_i, p_i, q_i, r_i, s_i, x_i, y_i, z_i, t_i, k_i, l_i, g_i, h_i / 1 \leq i \leq n\} \]
\[ \cup \{w_{n+1}, x_{n+1}, y_{n+1}\} \]
and the edge set
\[ E(G) = \{u_i v_i, u_i w_i, v_i w_{i+1}, u_i p_i, u_i q_i, w_i x_i, w_i y_i, v_i r_i, v_i s_i, u_i z_i, v_i t_i, z_i k_i, z_i l_i, t_i g_i, t_i h_i / 1 \leq i \leq n\} \]
\[ \cup \{v_n w_{n+1}, w_{n+1} x_{n+1}, w_{n+1} y_{n+1}\} \].

Define a function $f : V(G) \rightarrow \{1, 2, \ldots, q + 1\}$ by
\[ f(u_i) = 15i - 9 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(v_i) = 15i - 1 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(p_i) = 1 \]
\[ f(q_i) = 15i - 11 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(r_i) = 15i - 10 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(s_i) = 15i - 2 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(u_1) = 4 \]
\[ f(w_i) = 15i - 14 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(x_i) = 15i - 13 \quad \text{for} \quad 1 \leq i \leq n + 1 \]
\[ f(y_i) = 15i - 12 \quad \text{for} \quad 1 \leq i \leq n + 1 \]
\[ f(z_i) = 15i - 8 \quad \text{for} \quad 1 \leq i \leq n \]
Then the resulting edge labels are distinct. Thus $f$ provides a harmonic mean labeling of graph $G$.

Hence $G$ is a harmonic mean graph. \qed

**Example 3.** A harmonic mean labeling of graph $G$ obtained by $H$- super subdivision of path $HSS(P_5) \odot K_2$ are given in Figure 3.

![Figure 3](image)

**Theorem 2.4.** The $H$- super subdivision of path $HSS(P_n) \odot K_2$ is a harmonic mean graph.

**Proof.** Let $HSS(P_n) \odot K_2$ be the $H$- super subdivision of path graph whose vertex set

$$V(G) = \{u_i, v_i, w_i, p_i, q_i, r_i, s_i, x_i, y_i, z_i, t_i, k_i, l_i, g_i, h_i / 1 \leq i \leq n\}$$

and the edge set

$$E(G) = \{u_iv_i, u_iw_i, v_iw_{i+1}, u_iq_i, u_iq_i, p_iq_i, v_ir_i, v_is_i, r_is_i, w_ix_i, w_iy_i, x_iy_i, u_iz_i,$$

$$v_it_i, z_it_i, k_il_i, l_ig_i, t_ig_i, g_ig_i, h_i, h_i / 1 \leq i \leq n\} \cup \{v_{n+1}w_{n+1}, w_{n+1}x_{n+1}, w_{n+1}y_{n+1}\}$$. 
then the resultant graph is a harmonic mean labeling of $H$-super subdivision of path $HSS(P_n) \odot K_2$ graph.

Define a function $f : V(G) \to \{1, 2, ..., q + 1\}$ by

\[
\begin{align*}
  f(u_1) &= 8 \\
  f(u_i) &= 20i - 13 \quad \text{for} \quad 2 \leq i \leq n \\
  f(v_i) &= 20i - 1 \quad \text{for} \quad 1 \leq i \leq n \\
  f(w_1) &= 3 \\
  f(w_i) &= 20i - 19 \quad \text{for} \quad 2 \leq i \leq n \\
  f(p_i) &= 20i - 15 \quad \text{for} \quad 1 \leq i \leq n \\
  f(q_1) &= 7 \\
  f(q_i) &= 20i - 14 \quad \text{for} \quad 2 \leq i \leq n \\
  f(r_i) &= 20i - 3 \quad \text{for} \quad 1 \leq i \leq n \\
  f(s_i) &= 20i - 2 \quad \text{for} \quad 1 \leq i \leq n \\
  f(x_1) &= 1 \\
  f(x_i) &= 20i - 18 \quad \text{for} \quad 2 \leq i \leq n \\
  f(y_1) &= 2 \\
  f(y_i) &= 20i - 17 \quad \text{for} \quad 2 \leq i \leq n \\
  f(z_i) &= 20i - 11 \quad \text{for} \quad 1 \leq i \leq n \\
  f(t_i) &= 20i - 5 \quad \text{for} \quad 1 \leq i \leq n \\
  f(k_i) &= 20i - 10 \quad \text{for} \quad 1 \leq i \leq n \\
  f(l_i) &= 20i - 9 \quad \text{for} \quad 1 \leq i \leq n \\
  f(g_i) &= 20i - 8 \quad \text{for} \quad 1 \leq i \leq n \\
  f(h_i) &= 20i - 6 \quad \text{for} \quad 1 \leq i \leq n
\end{align*}
\]

Then the resulting edge labels are distinct. Thus $f$ provides a harmonic mean labeling of graph $G$.

Hence $G$ is a harmonic mean graph. \hfill \Box

\textbf{Example 4.} A harmonic mean labeling of graph $G$ obtained by $H$-super subdivision of path $HSS(P_4) \odot K_2$ are given in Figure 4.
3. Conclusion

We have presented four new results on Harmonic mean labeling of certain classes of graphs like the H-super subdivision of path $HSS(P_n)$, $HSS(P_n) \odot K_1$, $HSS(P_n) \odot \overline{K_2}$, $HSS(P_n) \odot K_2$. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.

References


DEPARTMENT OF MATHEMATICS
GOVERNMENT ARTS COLLEGE
C. Muttur, Chidambaram-608102, Tamil Nadu, India
E-mail address: meenasaravan14@gmail.com

DEPARTMENT OF MATHEMATICS
KRISHNASAMY COLLEGE OF SCIENCE ARTS AND MANAGEMENT FOR WOMEN
Cuddalore-607109, Tamil Nadu, India
E-mail address: sivasakthi2203@gmail.com