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EXACT SOLUTIONS TO THE COUPLED KLEIN-GORDON SYSTEM USING JACOBI ELLIPTIC FUNCTIONS

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ABSTRACT. Klein-Gordon equations are non linear partial differential equations representing several physical phenomena having many applications. There are several variants for Klein-Gordon equations. In this paper, a system of coupled Klein-Gordon equations are considered. Several new exact solutions for this system of equations are derived in this paper.

1. INTRODUCTION

The governing partial differential equations of many physical phenomena are non linear in in nature. Finding exact solutions to such equations are often difficult. But, exact solutions are necessary for the analysis of any physical problem. The available methods for finding exact solutions for nonlinear partial differential equations include inverse scattering method, Backlund transformation method, Lie group method, different ansatz methods and etc. [1–3, 5, 6, 11].

There are several variants for Klein-Gordon equations with many applications [6–8, 10, 11]. In this paper, we consider the system of coupled Klein-Gordon equations [9] given by

$$\begin{aligned} u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0, \\ v_x - v_t - 2(u^2)_t = 0, \end{aligned} \tag{1.1}$$

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where u and v are functions of the variables x and t. The exact solutions for this system of equations are derived in the following sections by assuming traveling wave solutions for this system. First of all we convert these equation in to an ordinary differential equation and then using Jacobi elliptic function ansatz method new exact solutions for this system of coupled equations are derived.

2. The method

We convert the given partial differential equations into an ordinary differential equations as follows. Using the transformation

$$u(t,x) = f(s) \quad \text{and} \quad v(t,x) = g(s), \tag{2.1}$$

where $s = at + bx$, the equations in (1.1) are converted into the following ordinary differential equations

$$b^2 f'' + 2fg + 2f^3 - a^2 f'' - f = 0 \quad \text{and} \quad (b-a)g' - 2a(f^2)' = 0. \tag{2.2}$$

Now, integrating the second equation given above, we get

$$g = \frac{2af^2}{b-a}. \tag{2.3}$$

Substituting this in equation (2.2), we get the following single ordinary differential equation corresponding to the system of coupled Klein-Gordon equations.

$$(a-b)^2(a+b)f'' + 2(a+b)f^3 + (a-b)f = 0.$$

We solve this equation in the next section to find the exact solutions the coupled Klein-Gordon equation. To derive the required solutions the ansatz form that we use is given by

$$f(s) = A_1\, G(s) + \frac{A_2}{G(s)},$$

where $G(s)$ is some ansatz function and A_1 and A_2 are arbitrary parameters to be determined.

3. Exact solutions

Exact solutions for the system of coupled Klein-Gordon equations (1.1) are derived in terms of Jacobi elliptic functions. Jacobi elliptic functions [4] are doubly periodic functions with modulus k, where $0 \leq k \leq 1$. For convenience we follow the parametric representation $e = k^2$ with $0 \leq e \leq 1$, for elliptic functions in this paper. The first ansatz we consider is

$$f(s) = A_1 \, \text{dn}(s) + \frac{A_2}{\text{dn}(s)}. \tag{3.1}$$

Substitute this ansatz in the equation (2.2). On simplification we get a sixth degree polynomial in dn(u), with even powers only. Equating to zero the coefficients of different powers of dn(u), we get the following non linear algebraic equations

$$2A_1(a+b)\left(A_1^2-(a-b)^2\right)=0, \quad A_2(a+b)\left((e+1)(a-b)^2+A_2^2\right)=0,$$
$$A_1\left(6A_1A_2(a+b)-(a-b)\left(a^2(e-2)-b^2(e-2)-1\right)\right)=0,$$
$$A_2\left(6A_1A_2(a+b)-(a-b)\left(a^2(e-2)-b^2(e-2)-1\right)\right)=0$$

Solving this system of non linear equations, we get the following sets of solutions

$$1) \quad e=\frac{\phi-6\psi}{\left(a^2-b^2\right)^2}, \quad A_1=a-b, \quad A_2=\frac{\psi}{(b-a)(a+b)^2}-3a+3b,$$
$$2) \quad e=\frac{\phi-6\psi}{\left(a^2-b^2\right)^2}, \quad A_1=b-a, \quad A_2=\frac{\psi}{(a-b)(a+b)^2}+3a-3b,$$
$$3) \quad e=\frac{6\psi+\phi}{\left(a^2-b^2\right)^2}, \quad A_1=a-b, \quad A_2=\frac{\psi}{(a-b)(a+b)^2}-3a+3b,$$
$$4) \quad e=\frac{6\psi+\phi}{\left(a^2-b^2\right)^2}, \quad A_1=b-a, \quad A_2=\frac{\psi}{(b-a)(a+b)^2}+3a-3b,$$

where $\phi=-16a^4+a^2\left(32b^2+1\right)-16b^4-b^2$ and $\psi=\sqrt{\left(b^2-a^2\right)^3\left(-8a^2+8b^2+1\right)}$.

Using these solutions, from equations (2.1), (2.3) and (3.1), we get the following two sets of exact solutions

$$u_1(t,x)=(a-b)\text{dn}\,(at+bx)+\left(\frac{\psi}{(b-a)(a+b)^2}-3a+3b\right)\frac{1}{\text{dn}\,(at+bx)},$$

and

$$u_2(t,x)=(b-a)\text{dn}\,(at+bx)+\left(\frac{\psi}{(a-b)(a+b)^2}+3a-3b\right)\frac{1}{\text{dn}\,(at+bx)},$$

where the value of the parameter of Jacobi elliptic function is

$$e = \frac{\phi - 6\psi}{\left(a^2 - b^2\right)^2}.$$

Second set of of solutions are

$$u_3(t,x) = (a-b)\text{dn}\,(at+bx) + \left(\frac{\psi}{(a-b)(a+b)^2} - 3a + 3b\right)\frac{1}{\text{dn}\,(at+bx)} \tag{3.2}$$

and

$$u_4(t,x) = (b-a)\text{dn}\,(at+bx) + \left(\frac{\psi}{(b-a)(a+b)^2} + 3a - 3b\right)\frac{1}{\text{dn}\,(at+bx)},$$

where the value of the parameter of the above Jacobi elliptic functions is

$$e = \frac{6\psi + \phi}{\left(a^2 - b^2\right)^2}.$$

In all the above solutions, the value of the second function v is given by the equation (2.3).

Second family of exact solutions are derived by assuming the solutions in the ansatz form

$$f(s) = A_1\,\text{cn}(s) + \frac{A_2}{\text{cn}(s)}.$$

Substitute this ansatz in (2.2). On simplification we get a sixth degree polynomial in cn(u), with even powers only. Equating to zero the coefficients of different powers of cn(u), we get the following non linear algebraic equations

$$2A_1(a+b)\left(A_1^2 - e(a-b)^2\right) = 0, \quad 2A_2(a+b)\left(A_2^2 - (e-1)(a-b)^2\right) = 0,$$
$$A_1\left((a-b)\left(a^2(2e-1) - 2b^2e + b^2 + 1\right) + 6A_1A_2(a+b)\right) = 0,$$
$$A_2\left((a-b)\left(a^2(2e-1) - 2b^2e + b^2 + 1\right) + 6A_1A_2(a+b)\right) = 0.$$

Solving this system of non linear equations, we get the following sets of solutions

$$1)\ e=\frac{\phi-3\psi}{16\left(a^2-b^2\right)^2},\ A_1=-\frac{1}{4}\sqrt{\frac{\phi-3\psi}{(a+b)^2}},\ A_2=\frac{\left(3\left(a^2-b^2\right)^2+\psi\right)\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4(a-b)(a+b)\left(a^2-b^2-1\right)}$$

$$2)\ e=\frac{\phi-3\psi}{16\left(a^2-b^2\right)^2},\ A_1=\frac{1}{4}\sqrt{\frac{\phi-3\psi}{(a+b)^2}},\ A_2=-\frac{\left(3\left(a^2-b^2\right)^2+\psi\right)\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4(a-b)(a+b)\left(a^2-b^2-1\right)}$$

$$3)\ e=\frac{3\psi+\phi}{16\left(a^2-b^2\right)^2},\ A_1=-\frac{1}{4}\sqrt{\frac{3\psi+\phi}{(a+b)^2}},\ A_2=\frac{\left(3\left(a^2-b^2\right)^2-\psi\right)\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4(a-b)(a+b)\left(a^2-b^2-1\right)}$$

$$4)\ e=\frac{3\psi+\phi}{16\left(a^2-b^2\right)^2},\ A_1=\frac{1}{4}\sqrt{\frac{3\psi+\phi}{(a+b)^2}},\ A_2=\frac{\left(\psi-3\left(a^2-b^2\right)^2\right)\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4(a-b)(a+b)\left(a^2-b^2-1\right)},$$

where

$$\phi=8a^4+a^2\left(1-16b^2\right)+8b^4-b^2 \quad \text{and} \quad \psi=\sqrt{(a-b)^2(a+b)^2\left(8\left(a^2-b^2\right)^2+1\right)}.$$

Using these solutions, from equations (2.1), (2.3) and (3.1), we get the following two sets of exact solutions

$$u_5(t,x)=-\frac{\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4}\text{cn}\,(at+bx)+\frac{\left(3\left(a^2-b^2\right)^2+\psi\right)\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4(a^2-b^2)\left(a^2-b^2-1\right)\text{cn}\,(at+bx)},$$

and

$$u_6(t,x)=\frac{\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4}\text{cn}\,(at+bx)-\frac{\left(3\left(a^2-b^2\right)^2+\psi\right)\sqrt{\frac{\phi-3\psi}{(a+b)^2}}}{4(a^2-b^2)\left(a^2-b^2-1\right)\text{cn}\,(at+bx)},$$

where the value of the parameter of Jacobi elliptic function is

$$e=\frac{\phi-3\psi}{16\left(a^2-b^2\right)^2}.$$

Second set of of solutions are

$$u_7(t,x)=-\frac{\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4}\text{cn}\,(at+bx)+\frac{\left(3\left(a^2-b^2\right)^2-\psi\right)\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4(a^2-b^2)\left(a^2-b^2-1\right)\text{cn}\,(at+bx)}$$

and

$$u_8(t,x)=\frac{\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4}\text{cn}\,(at+bx)+\frac{\left(\psi-3\left(a^2-b^2\right)^2\right)\sqrt{\frac{3\psi+\phi}{(a+b)^2}}}{4(a^2-b^2)\left(a^2-b^2-1\right)\text{cn}\,(at+bx)},$$

where the value of the parameter of the above Jacobi elliptic functions is

$$e = \frac{3\psi + \phi}{16\left(a^2 - b^2\right)^2}.$$

In all the above solutions, the value of the second function v is given by the equation (2.3).

4. Discussion

Two different families of traveling wave exact solutions for the system of coupled Klein-Gordon equations (1.1) are derived in this paper. The solutions are derived in terms of two different Jacobi elliptic functions $\text{dn}(s)$ and $\text{cn}(s)$. Similar ansatz forms can be used to derive other exact solutions for the coupled system of Klein-Gordon equations using the remaining Jacobi elliptic functions. It is also possible to derive soliton solutions from the derived new exact solutions. These solutions are obtained by letting the modulus of the Jacobi elliptic functions tends to one in the solutions. For example, choosing $a = \sqrt{b^2 - 1}$ in equation (3.2), the value of the parameter $e = 1$, and we get the soliton solution,

$$u(t, x) - \frac{\text{sech}\left(\sqrt{b^2 - 1}t + bx\right)}{\sqrt{b^2 - 1} + b}$$

and v is given by equation(2.3). Hence, we derived several doubly periodic and soliton exact solutions for the couple system of Klein-Gordon equations. The solutions obtained in terms of Jacobi elliptic functions are all new in the literature. Also, we have verified all the derived solutions using computer algebra system.

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