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MATRICES OVER MULTILATTICES

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ABSTRACT. This paper is an extension of the concept of Lattice matrices (L_n-matrices) to Multilattice matrices (M_n-matrices) using complete, consistent and distributive multilattice M with 0 and 1 along with some algebraic properties of these matrices are discussed.

1. INTRODUCTION

In the paper Lattice Matrices $(L_n - Matrices)$ [7], the author defined and explained the particular properties of algebra of square matrices over an arbitrary distributive lattice with 0 and 1 [1,3]. The properties of these matrices are useful tools in various situations like switching nets, automata theory, theory of finite graphs, etc. In this paper we extend the concept of Lattice matrices to Multilattice matrices($M_n - Matrices$)) using complete ,consistent and distributive multilattice M with 0 and 1 along with some algebraic properties of these matrices are discussed.

The organization of the paper is as follows. In section 2, the definition and some preliminary (theoretical)results about a multilattice are introduced. Later the concept of matrices over multilattices and Some definitions and properties of these matrices are introduced in section 3. In section 4, the concept of orthogonal matrices and some properties of these matrices are discussed.

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2. Preliminary Results

Let (M, $\leq$) is a partially ordered set and $U \subseteq M$, the multisupremum of U is a minimal element of the set of upper bounds of U and Multisup(U) denote the multisuprema of U. Dually we define the multiinfima.

Definition 2.1. *[2, 5, 6] A poset (M,$\leq$) be an ordered multilattice if and only if it satisfies the condition that for all $u, v, x$with $u \leq x$ and $v \leq x$, there exist $z \in \ Multisup \ \{u, v\}$ such that $z \leq x$ and its dual.*

When comparing with lattices, we see that least upper bound (which is a unique element) is replaced by the non empty set of all minimal (instead of least)upper bounds and dually.

Definition 2.2. *[5, 6] A multilattice is distributive if for each $u, v, w \in P$, the conditions $(u \vee v) \cap (u \vee w) \neq \emptyset$ and $(u \wedge v) \cap (u \wedge w) \neq \emptyset \Rightarrow v = w$ $(where \cap$ - is the usual set intersection and $\cup$ is the usual set unions). Similarly to lattice theory, if we define $(u \vee v) = Multisup\{u, v\}$ and $(u \wedge v) = Multiinf\{u, v\}$, then $(M, \wedge, \vee)$ be a algebraic multilattice and if we define $u \leq v$ if and only if $u \vee v = \{v\}$ and $u \wedge v = \{u\}$ it is possible to obtain the order version of multilattice.*

Definition 2.3. *A complete multilattice is a partially ordered set $(M, \leq)$ such that every subset $S \subseteq M$ the set of upper bounds of S has minimal (maximal) element, which are called multisuprema (multi infima), that is for any subset S of X, multiinf(S) and multiSup(S) exists and non empty.*

Definition 2.4. *Let(M, $\leq$) be a poset. The element $u \in M$ is called a greatest element of M if all other element are smaller. That is $u \geq x$ for every $x \in M$. similarly $v \in M$ is called a smallest element of M if $v \leq x$ for every $x \in M$.*

If a multilattice has a greatest element and smallest element, then (M, $\leq$) is said to be bounded. Normally greatest element is taken as 1 and smallest element is taken as 0.

Definition 2.5. *A multilattice M with 0 and 1 is called complemented if for each $u \in M$, there is atleast one element v such that $u \wedge v = \{0\}$ and $u \vee v = \{1\}$.*

Remark 2.1. *Let M be complete distributive multilattice. Then every element in M has exactly one complement in M.*

It is Noted that for any subset of a multilattice may not necessary have a supremum but a set of multisuprema. So we introduce some ordering between subsets of posets, which are the Hoare ordering, the Smyth ordering and the Egli-Milner ordering respectively.

Definition 2.6. *[4–6] consider $U, V \in 2^M$, then*

- $U \sqsubseteq_H V$ *if, and only if, for all* $a \in U$ *there exists* $b \in V$ *such that* $a \leq b$*;*
- $U \sqsubseteq_S V$ *if, and only if, for every* $b \in V$ *there exists* $a \in U$ *such that* $a \leq b$*,*
- $U \sqsubseteq_{EM} V$ *if, and only if,* $U \sqsubseteq_H V$ *and* $U \sqsubseteq_S V$*.*

Definition 2.7. *[4–6]* $(M, \wedge, \vee)$ *- be a algebraic multilattice . Let* $p \in M$ *and* U *and* V *be subsets of* M*, then*

- $p \wedge U = \cup \{(p \wedge x)/x \in U\}$*;*
- $p \vee U = \cup \{(p \vee x)/x \in U\}$*.*

Also, $U \wedge V = \cup\{(x \wedge y)/x \in U, y \in V\}$ *and* $U \vee V = \cup \{(x \vee y)/x \in U, y \in V\}$*.*

Throughout this paper, we use $U \leq V$ means $U \sqsubseteq_{EM} V$.

3. Matrices over Multilattices

Let M be a complete, consistent and distributive multilattice with 0 and 1. The multisup(p,q) is denoted by $p+q$ and multiinf(p,q) is denoted by $p.q$. Recall that multisuprimum and multiinfimum of elements are set of elements in M. In a lattice matrix [7] each entries of a matrix are single elements. Here we are taking a set of elements to each entry of a matrix from a multilattice M instead of taking a single elements. As defined in the lattice matrix , here we are defining matrices over a Multilattice along with some basic concepts and properties of these matrices are studied. In this chapter we use 0 and 1 for bottom and top element respectively in a multilattice M instead of using 0_M and 1_M.

Definition 3.1. *Let* M *be a complete, consistent and distributive multilattice with* 0 *and* 1*.The* $multisup(p, q)$ *is denoted by* $p+q$ *and* $multiinf(p, q)$ *is denoted by* $p.q$*. Let* M_n *(for* $n > o$*) be the set of* $n \times n$ *matrices over* M*, i.e.,* $M_n = \{P = (p_{ij})/p_{ij} \in 2^M\}$*, where* p_{ij} *is the* $(ij)^{th}$ *element of* P*.*

Definition 3.2. *Let* $P = (p_{ij})$*,* $Q = (q_{ij})$ *and* $R = (r_{ij})$ *are elements of* M_n*, we define:*

(1) $P + Q = P \vee_M Q = R$ *if, and only if,* $r_{ij} = p_{ij} + q_{ij}$*;*

(2) *$P \sqsubseteq_{EM} Q$ if, and only if, $p_{ij} \sqsubseteq_{EM} q_{ij}$;*
(3) *$P \wedge_M Q = R$ if, and only if, $r_{ij} = p_{ij}.q_{ij}$;*
(4) *$P.Q = PQ = R$ If and only $r_{ij} = \sum_{k=1}^{n} p_{ik}q_{kj}$;*
(5) *$P^T = R$ if, and only if, $r_{ij} = p_{ji}$;*
(6) *For $k \in M$, $kP = k.P = R$ if, and only if, $r_{ij} = k.p_{ij}$;*
(7) *$I = (p_{ij})$, where $p_{ij} = \{1\}$ for $i = j$*
and $= \{0\}$ for $i \neq j$;
(8) *$P^0 = I$,$P^{k+1} = P^k.P$;*
(9) *$O = (o_{ij})$,where $o_{ij} = 0$ for every i and j;*
(10) *$E = (e_{ij})$, where $e_{ij} = \{1\}$ for every i and j.*

Example 3.1. *Consider the multilattice in Figure 1.*

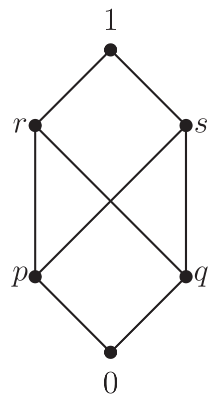

FIGURE 1. The multilattice in Example 3.1

Let

$$P = \begin{bmatrix} \{p\} & \{1\} \\ \{q\} & \{0\} \end{bmatrix}, Q = \begin{bmatrix} \{q\} & \{s\} \\ \{p\} & \{1\} \end{bmatrix}$$

$$P + Q = \begin{bmatrix} \{p+q\} & \{1+s\} \\ \{q+p\} & \{0+1\} \end{bmatrix} = \begin{bmatrix} \{r,s\} & \{1\} \\ \{r,s\} & \{1\} \end{bmatrix}$$

$$P \wedge_M Q = \begin{bmatrix} \{p.q\} & \{1.s\} \\ \{q.p\} & \{0.1\} \end{bmatrix} = \begin{bmatrix} \{0\} & \{s\} \\ \{0\} & \{0\} \end{bmatrix}$$

$$PQ = \begin{bmatrix} \{0+p\} & \{p+1\} \\ \{q+0\} & \{q+0\} \end{bmatrix} = \begin{bmatrix} \{p\} & \{1\} \\ \{q\} & \{q\} \end{bmatrix}$$

Properties with respect to addition and multiplication:

(1) $P + P \neq P$
(2) $P + Q = Q + P$
(3) $(P + Q) + R = P + (Q + R)$
(4) $PQ \neq QP$
(5) $(PQ)R = P(QR)$
(6) $P.I = I.P = P$
(7) $P.O = O.P = O$
(8) $P^i.A^j = A^{i+j}$
(9) $(P^i)^j = A^{ij}$
(10) $P(Q + R) = PQ + PR$
(11) $(P + Q)R = PR + QR$
(12) if $P \sqsubseteq_{EM} Q$ and $R \sqsubseteq_{EM} S$ then $PR \sqsubseteq_{EM} QS$
(13) Let $E = (e_{ij})$, where $e_{ij} = \{1\}$ for every i and j and
$I = (a_{ij})$, where $a_{ij} = \{1\}$ for $i = j$ and
$= \{0\}$ for $i \neq j$
Let $P = (p_{ij}$ be any matrix over a multilattice M.
Now if $I \sqsubseteq_{EM} P$ and $P \sqsubseteq_{EM} I$ then $I = P$.
Also if $P \sqsubseteq_{EM} E$ and $E \sqsubseteq_{EM} P$, then $E = P$.

Properties of transposition:

(1) $(P + q)^T = P^T + Q^T$
(2) if $P \sqsubseteq_{EM} Q$ then $P^T \sqsubseteq_{EM} Q^T$
(3) $(P \wedge_M Q)^T = P^T \wedge_M Q^T$
(4) $(P^T)^T = P$

Definition 3.3. *For $\alpha \in 2^M$ we shall use the notation $\alpha \hookrightarrow (P^m)_{ij}$, the ij^{th} entry of P^m whenever $\alpha = p_{i_0 i_1}.p_{i_1 i_2}.\cdots.p_{i_{m-1} i_m}$, where $i_0 = i$ and $i_m = j$ for some $i_1, i_2, \cdots, i_{m-1}$.*

Note 1. $(P^m)_{ij} = \sum_{\alpha \hookrightarrow (P^m)_{ij}} \alpha.$

Proposition 3.1. *If $\alpha \hookrightarrow (P^m)_{ij}$, where $m \geq n$, then there are integers k_1, k_2, k_3 and ν (all of them dependent on α) such that $0 \leq k_2 \leq n$, $k_1 + k_2 + k_3 = k$, $1 \leq \gamma \leq n$ and such that for each positive integer k:*

$$\alpha \sqsubseteq_{EM} (P^{k_1})_{i\gamma}.(P^{k.k_2})_{\gamma\gamma}.(P^{k_3})_{\gamma j}.$$

Proof. Let $\alpha = a_{i_0 i_1}.a_{i_1 i_2}.\cdots.a_{i_{m-1} i_m}$, where $\alpha \in 2^M$. Since $n \leq m$, Then $n \leq m+1$, two indices among the m+1 indices $i_0, i_1, \cdots, i_m$ must be equal. Let $i_s = i_t$, where $s < t$. Also, we can find such s and t such that $i_s = i_t, s < t$ and $t - s \leq n$. So, let $k_1 = s, k_2 = t - s, k_3 = m - t$ and $\nu = i_s = i_t$. □

Corollary 3.1. *If $\alpha \hookrightarrow (P^m)_{ij}$, where $m \geq n$, then there are natural numbers k_1, k_2, k_3 and γ such that $k_1 + k_2 \leq n$, $0 \leq k_2 \leq n$, $1 \leq \gamma \leq n$ and such that for each k,*

$$\alpha \sqsubseteq_{EM} (P^{k_1})_{i\gamma}.(P^{k.k_2})_{\gamma\gamma}.(P^{k_3})_{\gamma j}.$$

Theorem 3.1. *If $m \geq n$, then $(P^m)_{ij} \sqsubseteq_{EM} multisup(P^{m+(p.n!)})_{ij}$, where p is an arbitrary number.*

Proof. Suppose $\alpha \hookrightarrow (P^m)_{ij}$. Then by the above proposition, there are natural numbers k_1, k_2, k_3 and γ (all of them dependent on α) such that $0 < k_2 \leq n$, $k_1 + k_2 + k_3 = k$, $1 \leq \gamma \leq n$ and such that for each k, $\alpha \sqsubseteq_{EM} (P^{k_1})_{i\gamma}.((P^{k.k_1})_{\gamma\gamma}.(P^{k_2})_{\nu j}$.

Hence, $\alpha \sqsubseteq_{EM} (P^{k_1+k.k_2+k_3})_{ij} = (P^{m+(k-1).k_2})_{ij}$.

Replace $(k-1)$ by $(p.n!/k_2)$, where p is an arbitrary natural number.

Then $\alpha \sqsubseteq_{EM} (P^{m+(p.n!/k_2).k_2})_{ij} = (P^{m+pn!})_{ij}$, and further, for all $\alpha's$ such that $\sum_{\alpha \hookrightarrow (P^m)_{ij}} \alpha = (P^m)_{ij}$. Therefore, $\sum_{\alpha \hookrightarrow (P^m)_{ij}} \alpha \sqsubseteq_{EM} Multisup(P^{m+pn!})_{ij}$. This implies $(P^m)_{ij} \sqsubseteq_{EM} Multisup(P^{m+pn!})_{ij}$. □

4. Orthogonal Matrices

Definition 4.1. *A M_n Matrix P is called a unit if, and only if, there is an M_n matrix Q such that $PQ = QP = I$, and P is called orthogonal if, and only if, $PP^T = P^TP = I$.*

Proposition 4.1.

(1) *If $RQ = E$, then $EQ = E$.*
(2) *If $EPQ = E$, then $EQ = E$.*
(3) *Assume $P \wedge_M P = P$. Then $EP = E$ if, and only if, $I \sqsubseteq_{EM} P^TP$.*

Proof.

(1) For any matrix $EQ \sqsubseteq_{EM} E$ and $R \sqsubseteq_{EM} E$ are always true. Therefore by the property 12, $RQ \sqsubseteq_{EM} EQ$.

But $RQ = E$ implies $E \sqsubseteq_{EM} EQ$. Thus $EQ \sqsubseteq_{EM} E$ and $E \sqsubseteq_{EM} EQ$, this implies $E = Q$.

(2) This proof is a particular case of 1.

(3) Let $P \wedge_M P = P$.$EP = E$ holds if, and only if, for each i and j:

$$\{1\} = (EP)_{ij} = \sum_{k=1}^{n} e_{ik}p_{kj}$$
$$= \sum_{k=1}^{n} p_{kj} \text{ , since } e_{ik} = \{1\}$$
$$= \sum_{k=1}^{n} p_{kj}.p_{kj}$$
$$= \sum_{k=1}^{n} (P^T)_{jk}.P_{kj}$$
$$= (P^T.P)_{jj}, \text{ that is each diagonal entries are } \{1\}.$$

Hence, $EP = E$ holds if, and only if, $I \sqsubseteq_{EM} (P^T.P)$ holds.

□

Note 2. *From the above proposition we have ,$I \sqsubseteq_{EM} P^TP \Longrightarrow EP = E$, since $I \sqsubseteq_{EM} P^TP \Longrightarrow EI \sqsubseteq_{EM} EP^TP$. That is $I \sqsubseteq_{EM} P^TP$ implies $EP^TP = E$.*

Proposition 4.2. *If P is a unit, then P is orthogonal.*

Proof. If P is a unit, then there is a Q such that $PQ = QP = I$. This implies $Q^TP^T = P^TQ^T = I$. Hence, $E = EPQ = EQP = EQ^TP^T = EP^TQ^T$ and therefore by above proposition, we have $I \sqsubseteq_{EM} P^TP$,$I \sqsubseteq_{EM} PP^T$, $I \sqsubseteq_{EM} Q^TQ$, $I \sqsubseteq_{EM} QQ^T$. Then, to show That $P^TP \sqsubseteq_{EM} I$ and $PP^T \sqsubseteq_{EM} I$, that is to show that $P^TP \sqsubseteq_{EM} QP$ and $PP^T \sqsubseteq_{EM} PQ$ since $PQ = I$ and $QP = I$, it is suffices to show that $P^T \sqsubseteq_{EM} Q$ holds. But $I \sqsubseteq_{EM} Q^TQ \Longrightarrow P^T \sqsubseteq_{EM} P^TQ^TQ$, since $P^TQ^T = I$ and therefore $P^T \sqsubseteq_{EM} Q$ holds. Therefore, $P^TP \sqsubseteq_{EM} I$ and $PP^T \sqsubseteq_{EM} I$. This implies $P^TP = I$, P is orthogonal. □

Definition 4.2.

(1) *A set $\{M_1, M_2, \ldots, M_n\}$ of subsets of M is a decomposition of {1} in 2^M if, and only if, $\sum_{k=1}^{n} M_k = \{1\}$. That is $Multisup\{M_1, M_2, \ldots, M_n\} = \{1\}$.*

(2) *A set $\{M_1, M_2, \ldots, M_n\}$ of subsets of M is said to be orthogonal if, and only if, $M_iMj = \{0\}$. That is $multiinf\{Mi, Mj\} = 0$ provided $i \neq j$.*

(3) *A set of subsets of M is an orthogonal decomposition of $\{1\}$ in 2^M if, and only if, it is orthogonal and a decomposition of {1} in 2^M.*

We know that $I \sqsubseteq_{EM} P^TP$, $I \sqsubseteq_{EM} P^Tp$ implies $EP = E.$ Since P is orthogonal $PP^T = P^Tp = I$ implies $PP^T \sqsubseteq_{EM} I$, $P^TP \sqsubseteq_{EM} I$ $I \sqsubseteq_{EM} P^TP$ and $I \sqsubseteq_{EM} PP^T$. Also, $EP = E \Longrightarrow EP^T = E$. From this the following proposition follows.

Proposition 4.3. *A M_n is orthogonal if, and only if, each row and each column of it is an orthogonal decomposition of $\{1\}$ in 2^M.*

5. Conclusion

We have introduced the concept of Multilattices and also we defined and showed the algebraic properties of these matrices. Finally ,we observe that the extension of lattice Matrix to multilattice matrix is valid using a complete consistent distributive multilattice M.

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