ON TRIGONOMETRIC TOPOLOGICAL SPACES

S. MALATHI AND R. USHA PARAMESWARI

ABSTRACT. In this paper we introduce a new topological space, namely, Trigonometric topological space. A Strong trigonometric topological space is a topological space in which two topologies Sine and Cosine topologies induced from the given topology are coincide. Further, we discuss the properties of Interior and Closure operators in Sine and Cosine topological spaces.

1. INTRODUCTION

In this paper, we introduce Trigonometric topological spaces. These spaces are based on Sine and Cosine topologies. In a bitopological space we have considered two different topologies but in a trigonometric topological space the two topologies are derived from one topology. So, we observe that trigonometric topological space is different from bitopological space. Also, we define interior and closure operators in Sine and Cosine topological spaces and study their basic properties.

Section 2 deals with the preliminary concepts. In section 3, Sine and Cosine topologies are introduced together with their basic properties. The Trigonometric topological spaces are introduced in Section 4.

\[1\] Corresponding author

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2. Preliminaries

Throughout this paper $X$ denotes a set having elements from $[0, \pi/2]$. If $(X, \tau)$ is a topological space, then for any subset $A$ of $X$, $cl(A)$ denotes the closure of $A$, $int(A)$ denotes the interior of $A$. Further $X \setminus A$ denotes the complement of $A$ in $X$. The following definitions are very useful in the subsequent sections.

**Definition 2.1.** A topology on a set $X$ is a collection $\tau$ of subsets of $X$ having the following properties:

(i) $\emptyset, X$ are in $\tau$.

(ii) The union of elements of any subcollection of $\tau$ is in $\tau$.

(iii) The intersection of the elements of any finite subcollection of $\tau$ is in $\tau$.

The set $X$ together with the topology $\tau$ is called a topological space. The elements of $\tau$ are called open sets. The complement of an open set is called a closed set. The set of all closed sets in $X$ is denoted by $\tau^c$.

**Definition 2.2.** Let $X$ be a topological space. Let $A$ be a subset of $X$. Then the intersection of all closed sets containing $A$ is called the closure of $A$ and is denoted by $cl(A)$. Also, the union of all open sets contained in $A$ is called the interior of $A$ and is denoted by $int(A)$.

3. Sine and Cosine Topological Spaces

In this section, we introduce the concepts Sine and Cosine topological spaces and study their basic properties. Also, we discuss the properties of interior and closure operators in Sine and Cosine topological spaces. We begin this section by the construction of Sine topology.

**Construction of Sine Topology.** Let $X$ be any non-empty set having elements from $[0, \pi/2]$. Let $Sin X$ be the set consisting of the Sine values of the corresponding elements of $X$.

Define a function $f_s : X \rightarrow Sin X$ by $f_s(x) = Sin(x)$. Then $f_s$ is a bijective function. This implies, $f_s(\emptyset) = \emptyset$ and $f_s(X) = Sin X$. That is, $Sin \emptyset = \emptyset$.

**Result 3.1.** Let $X$ be a set and $A, B$ be subsets of $X$. Then $A \subseteq B$ if and only if $Sin A \subseteq Sin B$.

**Proof.** The proof is straight forward. $\square$
**Result 3.2.** The above result is not true for any subsets of $[0,2\pi)$. That is, $A=B$ implies $\sin A = \sin B$ is true for any subsets of $[0,2\pi)$. But $\sin A = \sin B$ does not imply $A=B$. For, $\sin\{0\} = \sin\{\pi\} = \{0\}$, but $\{0\} \neq \{\pi\}$. Hence $A=B$ iff $\sin A = \sin B$ is true only for $[0, \frac{\pi}{2}]$.

**Notation 3.1.** If $\tau$ is a topology on $X$, then $\tau_s$ denotes the set consisting of the images under $f_s$ of the corresponding elements of $\tau$.

**Result 3.3.** Let $(X, \tau)$ be a topological space. Then $\tau_s$ form a topology on $f_s(X)$.

**Proof.**

(i) Since $\emptyset$, $X \in \tau$, we have $f_s(\emptyset)$, $f_s(X) \in \tau_s$. That is, $\emptyset$, $f_s(X) \in \tau_s$.

(ii) Let $A_1, A_2, \ldots, A_n \in \tau_s$. Then $A_i = f_s(B_i)$, where $B_i \in \tau$ for $i = 1, 2, 3, \ldots$.

Since $\tau$ is a topology, we have $\bigcup_{i=1}^{\infty} B_i \in \tau$. This implies, $f_s\left(\bigcup_{i=1}^{\infty} B_i\right) \in \tau_s$.

That is, $\bigcup_{i=1}^{\infty} f_s(B_i) \in \tau_s$. Therefore, $\bigcup_{i=1}^{\infty} A_i \in \tau_s$.

(iii) Let $A_1, A_2, A_n \in \tau_s$. Then $A_i = f_s(B_i)$ where $B_i \in \tau$ for $i = 1, 2, \ldots, n$.

Since $\tau$ is a topology, we have $\bigcap_{i=1}^{n} B_i \in \tau$. This implies, $f_s\left(\bigcap_{i=1}^{n} B_i\right) \in \tau_s$.

That is, $\bigcap_{i=1}^{n} f_s(B_i) \in \tau_s$. Therefore, $\bigcap_{i=1}^{n} A_i \in \tau_s$. Hence $\tau_s$ is a topology on $f_s(X)$.

\qed

**Definition 3.1.** Let $(X, \tau)$ be a topological space. Then $\tau_s$ form a topology on $f_s(X)$. This topology is called a Sine topology of $X$. The space $(f_s(X), \tau_s)$ is said to be a Sine topological space corresponding to $X$.

That is, $\tau_s$ form a topology on $\sin X$. The elements of $\tau_s$ are called $\sin$-open sets. The complement of $\sin$-open set is said to be $\sin$-closed. The set of all $\sin$-closed subsets of $\sin X$ is denoted by $\tau$.

**Example 1.** Let $X = \{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}\}$ with topology $\tau = \{\emptyset, \{0\}, \{\frac{\pi}{3}\}, \{0, \frac{\pi}{3}\}, \{0, \frac{\pi}{4}\}, \{0, \frac{\pi}{3}, \frac{\pi}{4}\}, X\}$. Then $\sin X = \{0, \sqrt{3}, \frac{1}{2}, 1\}$, $\tau_s = \{\emptyset, \{0\}, \{\sqrt{3}\}, \{0, \sqrt{3}\}, \{0, \frac{1}{2}\}, \{0, \sqrt{3}, \frac{1}{2}\}, \sin X\}$.

Here the sets $\emptyset, \{0\}, \{\sqrt{3}\}, \{0, \sqrt{3}\}, \{0, \frac{1}{2}\}, \{0, \sqrt{3}, \frac{1}{2}\}, \sin X$ are called $\sin$-open sets & The $\sin$-closed sets are $\emptyset, \{1\}, \{\sqrt{3}, 1\}, \{\frac{1}{2}, 1\}, \{0, \sqrt{3}, 1\}, \{0, \frac{1}{2}, 1\}, \sin X$. 

**Construction of Cosine topology.** Let $\text{Cos}X$ be the set consisting of the Cosine values of the corresponding elements of $X$. Define a function $f_c : X \to \text{Cos}X$ by $f_c(x) = \text{Cos}x$. Then $f_c$ is bijective. Also, $f_c(\emptyset) = \emptyset$ and $f_c(X) = \text{Cos}X$. This implies, $\text{Cos}\emptyset = \emptyset$.

Let $\tau_{cs}$ be the set consisting of the images (under $f_c$) of the corresponding elements of $\tau$. Then we have,

$$\text{Cos}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \text{Cos}(A_n) \& \text{Cos}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \text{Cos}(A_n),$$

$$\text{Cos}X \setminus \text{Cos}A = \text{Cos}(X \setminus A),$$

$$A \subseteq B \iff \text{Cos}A \subseteq \text{Cos}B.$$

Using above facts, we can easily prove that $\tau_{cs}$ form a topology on $\text{Cos}X$. This topology is called Cosine topology (briefly, Cos-topology) of $X$. The pair $(\text{Cos}X, \tau_{cs})$ is called the Cosine topological space corresponding to $X$. The elements of $\tau_{cs}$ are called Cos-open sets. The complement of the Cos-open set is said to be Cos-closed. The set of all Cos-closed subsets of $\text{Cos}X$ is denoted by $\tau_c$.

**Example 2.** Let $X = \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology $\tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{3}\}, X\}$. Then $\text{Cos}X = \{1, \frac{1}{\sqrt{2}}, 0\}, \tau_{cs} = \{\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, \text{Cos}X\}$. Here the subsets $\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, \text{Cos}X$ are Cos-open sets.

The Cos-closed sets are $\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, \text{Cos}X$.

That is, $\tau_{cs} = \{\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, \text{Cos}X\}$.

**Definition 3.2.** The topology $\tau$ on $X$ is said to be a Strong trigonometric topology if its Sine and Cosine topologies are coincide. That is, if $\tau_s = \tau_{cs}$, then $\tau$ is said to be a Strong trigonometric topology. The space $X$ together with this $\tau$ is called a Strong trigonometric topological space.

**Definition 3.3.** Let $X$ be a set having elements from $[0, \frac{\pi}{2}]$ and $\tau$ be the topology on $X$. If $\tau_s$ and $\tau_{cs}$ are exist and unequal, then $\tau$ is said to be Weak trigonometric topology and the space $(X, \tau)$ is called a Weak trigonometric topological space.

**Example 2.** Let $X = \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology

$\tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{3}\}, \{\frac{\pi}{3}, \frac{\pi}{2}\}, X\}.$

Then $\text{Sin}X = \text{Cos}X = \{0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$. Also, $\tau_s = \tau_{cs}$. This implies, $\tau$ is a Strong trigonometric topology. Hence $(X, \tau)$ is a Strong trigonometric topological space.
Let \( X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\} \). Then clearly, \( \tau = \{\emptyset, \{\frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}\}, X\} \) is a topology on \( X \). Here \( \sin X = \cos X \) but \( \tau_s \) and \( \tau_{cs} \) are unequal. Therefore, \( \tau \) is a Weak trigonometric topology and the space \((X, \tau)\) is a Weak trigonometric topological space.

**Remark 3.1.** Every Strong trigonometric topology (resp. Weak trigonometric topology) is a topology but the converse is not true. The above two examples proves this. From this, we observe that the topological space is either a Strong trigonometric topological space or a Weak trigonometric topological space. The following results are true for Strong and Weak trigonometric topological spaces. But both are a topological space and a topological space is either of that. So, we simply write \((X, \tau)\) is a topological space in the following results.

**Definition 3.4.** Let \((X, \tau)\) be a topological space and \( A \subseteq \sin X \). The union of all \( \sin \)-open sets contained in \( A \) is called a Sine-interior of \( A \) and it is denoted by \( \text{Int}_{\sin}(A) \). Also, the intersection of all \( \sin \)-closed containing \( A \) is called a Sine-closure of \( A \) and it is denoted by \( \text{Cl}_{\sin}(A) \). That is,

\[
\text{Int}_{\sin}(A) = \bigcup \{B \subseteq \sin X : B \subseteq A \text{ & } B \text{ is } \sin \text{-open}\}
\]

\[
\text{Cl}_{\sin}(A) = \bigcap \{B \subseteq \sin X : A \subseteq B \text{ & } B \text{ is } \sin \text{-closed}\}.
\]

**Result 3.4.** Let \((X, \tau)\) be a topological space and \( A \subseteq \sin X \). Then \( \text{Int}_{\sin}(A) \) is a \( \sin \)-open set.

**Proof.** The proof follows from the fact that the union of any collection of \( \sin \)-open sets is \( \sin \)-open. \(\square\)

**Result 3.5.** Let \((X, \tau)\) be a topological space and \( A \subseteq \sin X \). Then \( \text{Int}_{\sin}(A) \subseteq A \).

**Proof.** Let \( x \in \text{Int}_{\sin}(A) \). Then \( x \in B \) for some \( \sin \)-open set \( B \subseteq A \). This implies, \( x \in A \). Therefore, \( \text{Int}_{\sin}(A) \subseteq A \). \(\square\)

**Result 3.6.** Let \((X, \tau)\) be a topological space and \( A \subseteq \sin X \). Then \( \text{Int}_{\sin}(A) \) is the largest \( \sin \)-open set contained in \( A \) & \( A \) is \( \sin \)-open if and only if \( A = \text{Int}_{\sin}(A) \).

**Proof.** It follows directly from the definition and Result 3.5. \(\square\)

**Result 3.7.** Let \((X, \tau)\) be a topological space and \( A, B \) be subsets of \( \sin X \). Then

\( A \subseteq B \Rightarrow \text{Int}_{\sin}(A) \subseteq \text{Int}_{\sin}(B) \), \( \text{Int}_{\sin}(A \cap B) = \text{Int}_{\sin}(A) \cap \text{Int}_{\sin}(B) \) & \( \text{Int}_{\sin}(A) \cup \text{Int}_{\sin}(B) \subseteq \text{Int}_{\sin}(A \cup B) \).
Proof. It is obvious. \hfill \square

**Remark 3.2.** \( \text{Int}_{\sin}(A \cup B) \) need not be equal to \( \text{Int}_{\sin}(A) \cup \text{Int}_{\sin}(B) \).

For example, let \( X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2} \} \) with topology \( \tau = \{\emptyset, \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}, X\} \).

Then \( \text{Int}_{\sin}X = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\} \) and 

\[ \tau_s = \{0, \{0, \frac{1}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \text{Sin}X\}. \]

Let \( A = \{0\} \) and \( B = \{\frac{1}{2}\} \). Then \( A \cup B = \{0, \frac{1}{2}\} \). Now, \( \text{Int}_{\sin}(A) = \emptyset \), \( \text{Int}_{\sin}(B) = \emptyset \). This implies, \( \text{Int}_{\sin}(A) \cup \text{Int}_{\sin}(B) = \emptyset \). Also, \( \text{Int}_{\sin}(A \cup B) = \{0, \frac{1}{2}\} \).

Therefore, \( \text{Int}_{\sin}(A \cup B) \neq \text{Int}_{\sin}(A) \cup \text{Int}_{\sin}(B) \).

**Result 3.8.** Let \( (X, \tau) \) be a topological space and \( A \subseteq \text{Sin}X \). Then \( \text{Cl}_{\sin}(A) \) is a Sin-closed set, \( A \subseteq \text{Cl}_{\sin}(A) \), \( \text{Cl}_{\sin}(A) \) is the smallest Sin-closed set containing \( A \). \( A \) is Sin-closed if and only if \( A = \text{Cl}_{\sin}(A) \).

**Result 3.9.** Let \( (X, \tau) \) be a topological space and \( A, B \) be subsets of \( \text{Sin}X \). If \( A \subseteq B \), then \( \text{Cl}_{\sin}(A) \subseteq \text{Cl}_{\sin}(B) \).

**Result 3.10.** Let \( (X, \tau) \) be a topological space and \( A, B \) be subsets of \( \text{Sin}X \). Then

\[
\text{Cl}_{\sin}(A \cup B) = \text{Cl}_{\sin}(A) \cup \text{Cl}_{\sin}(B),
\]

\[
\text{Cl}_{\sin}(A \cap B) \subseteq \text{Cl}_{\sin}(A) \cap \text{Cl}_{\sin}(B).
\]

**Remark 3.3.** \( \text{Cl}_{\sin}(A \cap B) \) need not be equal to \( \text{Cl}_{\sin}(A) \cap \text{Cl}_{\sin}(B) \).

For example, let \( X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2} \} \) with topology \( \tau = \{\emptyset, \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}, X\} \).

Then \( \text{Sin}X = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\} \) and 

\[ \tau_s = \{0, \{0, \frac{1}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \text{Sin}X\}. \]

This implies, \( \tau_s^c = \{0, \{1\}, \{0, \frac{1}{2}, 1\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}, \text{Sin}X\} \).

Let \( A = \{0, \frac{1}{2}\} \) and \( B = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\} \). Then \( A \cap B = \emptyset \). Now, \( \text{Cl}_{\sin}(A) = \{0, \frac{1}{2}, 1\} \), \( \text{Cl}_{\sin}(B) = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\} \).

This implies, \( \text{Cl}_{\sin}(A) \cap \text{Cl}_{\sin}(B) = \{1\} \). Also, \( \text{Cl}_{\sin}(A \cap B) = \emptyset \). Therefore, \( \text{Cl}_{\sin}(A \cap B) \neq \text{Cl}_{\sin}(A) \cap \text{Cl}_{\sin}(B) \).

**Result 3.11.** Let \( (X, \tau) \) be a topological space and \( A \subseteq X \).

Then \( \text{Sin} (\text{int}(A)) = \text{Int}_{\sin}(\text{Sin}A), \text{Cl}_{\sin}(\text{Sin}A) = \text{Sin} (\text{Cl}(A)) \) & \( \text{Sin}X \setminus (\text{Int}_{\sin}(\text{Sin}A)) = \text{Cl}_{\sin}(\text{Sin}X \setminus \text{Sin}A) \).
Definition 3.5. Let \((X, \tau)\) be a topological space and \(A \subseteq \text{Cos} X\). Then we define,

\[
\text{Int}_{\text{cos}}(A) = \bigcup \{ B \subseteq \text{Cos} X : B \subseteq A \text{ and } B \text{ is Cos - open} \}
\]

\[
\text{Cl}_{\text{cos}}(A) = \bigcap \{ B \subseteq \text{Cos} X : A \subseteq B \text{ and } B \text{ is Cos - closed} \}.
\]

The proof of the following result follows directly from the definition.

Result 3.12. Let \((X, \tau)\) be a topological space and \(A, B\) be subsets of \(\text{Cos} X\). Then

(i) \(\text{Int}_{\text{cos}}(A)\) is a Cos-open set
(ii) \(\text{Int}_{\text{cos}}(A) \subseteq A\)
(iii) \(\text{Int}_{\text{cos}}(A)\) is the largest Cos-open set contained in \(A\)
(iv) \(A\) is Cos-open if and only if \(A = \text{int}_{\text{cos}}(A)\)
(v) \(A \subseteq B \Rightarrow \text{Int}_{\text{cos}}(A) \subseteq \text{Int}_{\text{cos}}(B)\)
(vi) \(\text{Int}_{\text{cos}}(A \cap B) = \text{Int}_{\text{cos}}(A) \cap \text{Int}_{\text{cos}}(B)\)
(vii) \(\text{Int}_{\text{cos}}(A) \cup \text{Int}_{\text{cos}}(B) \subseteq \text{Int}_{\text{cos}}(A \cup B)\).

The equality does not hold in (vii).

For example, let \(X = \{0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}\) with topology \(\tau = \{\emptyset, \{0\}, \{\frac{\pi}{4}\}, \{0, \frac{\pi}{4}\}, \{0, \frac{\pi}{3}, \frac{\pi}{4}\}, X\}\).

Then \(\text{Cos} X = \{1, \frac{1}{\sqrt{2}}, 0\}\) and

\(\tau_{\text{cos}} = \{\emptyset, \{1\}, \{\frac{1}{\sqrt{2}}\}, \{1, \frac{1}{\sqrt{2}}\}, \{1, \frac{1}{\sqrt{2}}, \frac{1}{2}\}, \text{Cos} X\}\).

Consider the subsets \(A = \{1\}\) and \(B = \{\frac{1}{\sqrt{2}}\}\). Then \(A \cup B = \{1, \frac{1}{\sqrt{2}}\}\). Now, \(\text{Int}_{\text{cos}}(A) = \{1\}\) and \(\text{Int}_{\text{cos}}(B) = \emptyset\). This implies, \(\text{Int}_{\text{cos}}(A) \cup \text{Int}_{\text{cos}}(B) = \{1\}\).

Also, \(\text{Int}_{\text{cos}}(A \cup B) = \{1, \frac{1}{\sqrt{2}}\}\).

Therefore, \(\text{Int}_{\text{cos}}(A \cup B) \neq \text{Int}_{\text{cos}}(A) \cup \text{Int}_{\text{cos}}(B)\).

Result 3.13. Let \((X, \tau)\) be a topological space and \(A, B\) be subsets of \(\text{Cos} X\). Then

(i) \(\text{Cl}_{\text{cos}}(A)\) is a Cos-closed set
(ii) \(A \subseteq \text{Cl}_{\text{cos}}(A)\)
(iii) \(\text{Cl}_{\text{cos}}(A)\) is the smallest Cos-closed set containing \(A\)
(iv) \(A\) is Cos-closed if and only if \(A = \text{Cl}_{\text{cos}}(A)\)
(v) \(A \subseteq B \Rightarrow \text{Cl}_{\text{cos}}(A) \subseteq \text{Cl}_{\text{cos}}(B)\)
(vi) \(\text{Cl}_{\text{cos}}(A \cup B) = \text{Cl}_{\text{cos}}(A) \cup \text{Cl}_{\text{cos}}(B)\)
(vii) \(\text{Cl}_{\text{cos}}(A \cap B) \subseteq \text{Cl}_{\text{cos}}(A) \cap \text{Cl}_{\text{cos}}(B)\).

Remark 3.4. The reverse inclusion of (vii) in above result need not be true, which can be verified from the following example.
Example 3. Let $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology $\tau = \{\emptyset, \{0, \frac{\pi}{6}\}, \{0, \frac{\pi}{6}, \frac{\pi}{4}\}, \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\}, X\}$. Then $\cos X = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\} \text{ and } \tau_{\cos} = \{\emptyset, \{\frac{\sqrt{3}}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}, \cos X\}$. This implies, $\tau_{\cos}^c = \{\emptyset, \{0\}, \{0, \frac{\sqrt{3}}{2}, 1\}, \{0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}, \cos X\}$.

Consider the subsets $A = \{\frac{\sqrt{3}}{2}\}$ and $B = \{1, \frac{1}{\sqrt{2}}\}$. Then $A \cap B = \emptyset$. Now, $\text{Cl}_{\cos}(A) = \{0, \frac{\sqrt{3}}{2}, 1\}, \text{Cl}_{\cos}(B) = \cos X$. This implies, $\text{Cl}_{\cos}(A) \cap \text{Cl}_{\cos}(B) = \{0, \frac{\sqrt{3}}{2}, 1\}$ Also, $\text{Cl}_{\cos}(A \cap B) = \emptyset$. Therefore, $\text{Cl}_{\cos}(A \cap B) \neq \text{Cl}_{\cos}(A) \cap \text{Cl}_{\cos}(B)$.

Result 3.14. Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then

(i) $\cos(\text{int}(A)) = \text{Int}_{\cos}(\cos A)$
(ii) $\text{Cl}_{\cos}(\cos A) = \cos(\text{cl}(A))$
(iii) $\cos X \setminus (\text{Int}_{\cos}(\cos A)) = \text{Cl}_{\cos}(\cos X \setminus \cos A)$.

Proof. It is obvious. 

4. Trigonometric topological spaces

In this section, we study about Trigonometric topological spaces. Also, we discuss about some set theory relations.

Definition 4.1. Let $X$ be a set having elements from $[0, \frac{\pi}{2}]$. Define $T_u(X)$ by $T_u(X) = \sin X \cup \cos X$.

Result 4.1. Let $X$ be a set and $A$ be a subset of $X$. Then

(i) $T_u(X) \setminus (\sin A \cup \cos A) = (T_u(X) \setminus \sin A) \cap (T_u(X) \setminus \cos A)$,
(ii) $T_u(X) \setminus (\sin A \cap \cos A) = (T_u(X) \setminus \sin A) \cup (T_u(X) \setminus \cos A)$.

Proof. The proof follows directly from De-Morgan’s law. 

Result 4.2. Let $X$ be a set and $A$ be a subset of $X$. Then

(i) $T_u(X) \setminus (\sin A \cap \cos A) = (\sin X \setminus \sin A) \cup (\cos X \setminus \cos A)$,
(ii) $\sin X \setminus A \subseteq T_u(X) \setminus \sin A$,
(iii) $\cos X \setminus A \subseteq T_u(X) \setminus \cos A$,
(iv) $T_u(X) \setminus \sin A = (\sin X \setminus \sin A) \cup (\cos X \setminus \sin A)$,
(v) $T_u(X) \setminus \cos A = (\sin X \setminus \cos A) \cup (\cos X \setminus \sin A)$. 

Result 4.3. From Result 4.1 and 4.2, we observe that \((T_u(X) \setminus \sin A) \cup (T_u(X) \setminus \cos A)\)  
\[= (\sin X \setminus \sin A) \cup (\cos X \setminus \cos A). \]  
But \((T_u(X) \setminus \sin A) \neq (\sin X \setminus \sin A)\) and \((T_u(X) \setminus \cos A) \neq (\cos X \setminus \cos A). \) Also, from (iv), \(T_u(X) \setminus \sin X = \cos X \setminus \sin X\) and from (v), \(T_u(X) \setminus \cos X = \sin X \setminus \cos X\).

Result 4.4. Let \(X\) be a set and \(A \subseteq X\). Then \(T_u(X) \setminus (\sin A \cup \cos A) \subseteq (\sin X \setminus \sin A) \cup (\cos X \setminus \cos A)\).

The equality does not hold. For example, let \(X = \{0, \frac{\pi}{3}, \frac{\pi}{2}\}\). Then \(\sin X = \{0, \frac{\sqrt{3}}{2}, 1\} \) & \(\cos X = \{1, \frac{1}{2}, 0\}\). Consider the subset \(A = \{0\}\). Then \(T_u(X) \setminus (\sin A \cup \cos A) = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}\) and \((\sin X \setminus \sin A) \cup (\cos X \setminus \cos A) = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}\). Therefore, \(T_u(X) \setminus (\sin A \cup \cos A) \neq (\sin X \setminus \sin A) \cup (\cos X \setminus \cos A)\).

Definition 4.2. Let \(X\) be a set having elements from \([0, \frac{\pi}{2}]\). Define \(T_i(X)\) by \(T_i(X) = \sin X \cap \cos X\).

Result 4.5. Let \(X\) be a set and \(A\) be a subset of \(X\). Then

(i) \(T_i(X) \setminus (\sin A \cup \cos A) = (T_i(X) \setminus \sin A) \cap (T_i(X) \setminus \cos A)\),
(ii) \(T_i(X) \setminus (\sin A \cup \cos A) = (\sin X \setminus \sin A) \cap (\cos X \setminus \cos A)\),
(iii) \(T_i(X) \setminus (\sin A \cup \cos A) = (T_i(X) \setminus \sin A) \cup (T_i(X) \setminus \cos A)\),
(iv) \(T_i(X) \setminus \sin A \subseteq \sin X \setminus A\),
(v) \(T_i(X) \setminus \cos A \subseteq \cos X \setminus A\),
(vi) \(T_i(X) \setminus \sin A = (\sin X \setminus \sin A) \cap (\cos X \setminus \sin A)\),
(vii) \(T_i(X) \setminus \cos A = (\sin X \setminus \cos A) \cap (\cos X \setminus \cos A)\).

Remark 4.1. The reverse inclusion of (iv) need not be true.

For example, let \(X = \{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}\}\). Then \(\sin X = \{0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 1\}\) & \(\cos X = \{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1\}\). Also, \(T_i(X) = \{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1\}\). Let \(A = \{\frac{\pi}{1}, \frac{\pi}{2}\}\). Then \((\sin X \setminus \sin A) \cup (\cos X \setminus \cos A) \nsubseteq T_i(X) \setminus (\sin A \cap \cos A)\).

Note 1. Let \(X\) be a set having elements from \([0, \frac{\pi}{2}]\). Then
\[T_i(X) \setminus \sin X = \emptyset \text{ and } T_i(X) \setminus \cos X = \emptyset \text{ and} \]
\[T_u(X) \setminus T_i(X) = (\cos X \setminus \sin X) \cup (\sin X \setminus \cos X). \]

Definition 4.3. Let \(X\) be a set with elements from \([0, \frac{\pi}{2}]\) and \(\tau\) be the topology on \(X\). We define a set \(J = \{\emptyset, U \cup V \cup T_i(X) : U \in \tau_s \text{ and } V \in \tau_{cs}\}\). Then \(J\) form a
topology on \( T_u(X) \). This topology is called trigonometric topology on \( T_u(X) \). The pair \( (T_u(X), \mathcal{J}) \) is called a trigonometric topological space.

The elements of \( \mathcal{J} \) are called trigonometric open sets. The complement of a trigonometric open set is said to be a trigonometric closed set. The set of all trigonometric closed sets is denoted by \( \mathcal{J}^c \).

**Example 4.** Let \( X = \left\{ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3} \right\} \) with topology \( \tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{4}, \frac{\pi}{3}\}, X\} \). Then \( \tau_s = \{\emptyset, \{\frac{1}{2}\}, \{1, \frac{1}{\sqrt{2}}\}, \operatorname{Sin}X\} \) and \( \tau_{cs} = \{\emptyset, \{\frac{\sqrt{3}}{2}\}, \{0, 1, \frac{1}{\sqrt{2}}\}, \operatorname{Cos}X\} \). Now, \( \mathcal{J} = \{\emptyset, \operatorname{T}_i(X), \operatorname{Cos}X, \left\{\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right\}, \{0, \frac{1}{\sqrt{2}}, 1, \frac{\sqrt{3}}{2}\}, \{1, \frac{1}{\sqrt{2}}, 1, 0\}, \{\frac{1}{2}, \frac{1}{\sqrt{2}}\}, \{0, \frac{1}{\sqrt{2}}, \frac{1}{2}\}, \{1, 0, \frac{\sqrt{3}}{2}\}, \{1, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{1, 0, \frac{1}{\sqrt{2}}\}, \operatorname{Sin}X, T_u(X)\} \). This implies, \( \mathcal{J} \) is a topology on \( T_u(X) \).

**Remark 4.2.** If \( \mathcal{J} \) is a trigonometric topology, then \( \mathcal{J} \) must contain the elements \( \emptyset, \operatorname{T}_i(X), \operatorname{Sin}X, \operatorname{Cos}X, T_u(X) \). For (i) clearly by definition, \( \emptyset \in \mathcal{J} \), (ii) Since \( \emptyset \) is open in both \( \operatorname{Sin}X \) and \( \operatorname{Cos}X \), we have \( \emptyset \cup \emptyset \cup \operatorname{T}_i(X) = \operatorname{T}_i(X) \in \mathcal{J} \), (iii) Since \( \emptyset \) is open in \( \operatorname{Cos}X \) and \( \operatorname{Sin}X \) is open in \( \operatorname{Sin}X \), \( \operatorname{Sin}X \cup \emptyset \cup \operatorname{T}_i(X) = \operatorname{Sin}X \in \mathcal{J} \), (iv) Since \( \emptyset \) is open in \( \operatorname{Sin}X \) and \( \operatorname{Cos}X \) is open in \( \operatorname{Cos}X \), \( \emptyset \cup \operatorname{Cos}X \cup \operatorname{T}_i(X) = \operatorname{Cos}X \in \mathcal{J} \) & (v) \( \operatorname{Sin}X \cup \operatorname{Cos}X \cup \operatorname{T}_i(X) = \operatorname{Sin}X \cup \operatorname{Cos}X = T_u(X) \in \mathcal{J} \). Also, the smallest non-empty set as an element of \( \mathcal{J} \) is \( \operatorname{T}_i(X) \).

**Definition 4.4.** Let \( (X, \tau) \) be a topological space with \( \operatorname{Sin}X \neq \operatorname{Cos}X \). If \( \mathcal{J} \) contains only the elements \( \emptyset, \operatorname{T}_i(X), \operatorname{Sin}X, \operatorname{Cos}X, T_u(X) \) then \( \mathcal{J} \) is said to be a Standard trigonometric topology. It is denoted by the symbol \( \mathcal{J}_s \). The pair \( (T_u(X), \mathcal{J}_s) \) is called a Standard trigonometric topological space.

**Note 2.** Trigonometric topology is finer than the Standard trigonometric topology on \( T_u(X) \). That is, \( \mathcal{J}_s \subseteq \mathcal{J} \).

**Theorem 4.1.** Let \( (X, \tau) \) be any topological space with \( \operatorname{Sin}X = \operatorname{Cos}X \). Then \( \mathcal{J} \) is an indiscrete topology on \( T_u(X) \).

*Proof.* Given that \( \operatorname{Sin}X = \operatorname{Cos}X \). Then \( \operatorname{T}_i(X) = T_u(X) = \operatorname{Sin}X = \operatorname{Cos}X \).

Also, \( \operatorname{T}_i(X) \) is the smallest non-empty set in \( \mathcal{J} \).

This implies, \( \mathcal{J} = \{\emptyset, T_u(X)\} \). Therefore, \( \mathcal{J} \) is an indiscrete topology on \( T_u(X) \). \( \square \)

The proof of the following Theorem follows from Theorem 4.1

**Theorem 4.2.** If \( \tau \) is a Strong trigonometric topology, then \( \mathcal{J} \) is an indiscrete topology.
Theorem 4.3. If $(X, \tau)$ be an indiscrete topological space with $\text{Sin}X \neq \text{Cos}X$, then $\mathcal{J}$ is a standard trigonometric topology.

Theorem 4.4. If $T_i(X) = \text{Sin}X \setminus \{p\}$ where $p \in \text{Sin}X$, then $\mathcal{J}$ is a Standard trigonometric topology.

Theorem 4.5. If $T_i(X) = \text{Cos}X \setminus \{q\}$ where $q \in \text{Cos}X$, then $\mathcal{J}$ is a Standard trigonometric topology.

Theorem 4.6. If $T_i(X) = T_u(X) \setminus \{p, q\}$ where $p, q \in T_u(X)$, then $\mathcal{J}$ is a Standard trigonometric topology.

Proof. Assume that $p, q \notin T_i(X)$.

If $p, q \in \text{Sin}X$, then $T_i(X) = \text{Cos}X$. This implies, number of elements of $\text{Sin}X$ is two more than $\text{Cos}X$. This is a contradiction. Therefore both $p$ and $q$ does not belongs to $\text{Sin}X$. Similarly, we can prove both $p$ and $q$ does not belongs to $\text{Cos}X$.

Hence, either $p \in \text{Sin}X$ and $q \in \text{Cos}X$ (or) $q \in \text{Sin}X$ and $p \in \text{Cos}X$. Therefore, the proof follows from Theorem 4.4 and Theorem 4.5. □

5. Conclusion

In this paper, we have introduced Sine and Cosine topological spaces and studied their basic properties. Also, we have introduced Trigonometric topological spaces. Further, we have discussed the properties of interior and closure operators in Sine and Cosine topologies.

References


DEPARTMENT OF MATHEMATICS
GOVINDAMMAL ADITANAR COLLEGE FOR WOMEN
TIRUCHENDUR
E-mail address: malathis2795@gmail.com

DEPARTMENT OF MATHEMATICS
GOVINDAMMAL ADITANAR COLLEGE FOR WOMEN
TIRUCHENDUR-628 215
E-mail address: rushaparamesvari@gmail.com