DECOMPOSITIONS OF $\pi g$-CONTINUITY VIA IDEAL NANO TOPOLOGICAL SPACES

O. NETHAJI$^1$, R. ASOKAN, AND I. RAJASEKARAN

ABSTRACT. In this paper, we introduce and discuss some notions of $I_{n\pi g}$-closed sets, $I_{n\pi g}$-continuity in ideal nano spaces.

1. INTRODUCTION AND PRELIMINARIES

According to [14], an ideal $I$ on a topological space $(X, \tau)$ is a non-empty collection of subsets of $X$ which satisfies the following conditions.

(i) $A \in I$ and $B \subseteq A$ imply $B \in I$ and
(ii) $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a topological space $(X, \tau)$ with an ideal $I$ on $X$. If $\wp(X)$ is the family of all subsets of $X$, a set operator $(\cdot)^* : \wp(X) \to \wp(X)$, called a local function of $A$ with respect to $\tau$ and $I$ is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$, [3].

The closure operator defined by $cl^*(A) = A \cup A^*(I, \tau)$, [13] is a Kuratowski closure operator which generates a topology $\tau^*(I, \tau)$ called the $*$-topology finer than $\tau$. The topological space together with an ideal on $X$ is called an ideal topological space or an ideal space denoted by $(X, \tau, I)$. We will simply write $A^*$ for $A^*(I, \tau)$ and $\tau^*$ for $\tau^*(I, \tau)$.

$^1$corresponding author

2010 Mathematics Subject Classification. 54A05, 54A10, 54C08, 54C10.

Key words and phrases. nano $\pi g$-closed sets, $I_{n\pi g}$-closed sets and $I_{n\pi g}$-continuity.
In this paper, we introduce and discuss some notions of \( I_{n\pi g}\)-closed sets, \( I_{\pi g}\)-continuity in ideal nano spaces.

We denote a nano topological space by \((U, \mathcal{N})\), where \( \mathcal{N} = \tau_R(X) \). The nano-interior, nano-closure and nano \( \alpha\)-closure of a subset \( A \) of \( U \) are denoted by \( I_n(A), C_n(A) \) and \( C_{n\alpha}(A) \), respectively.

An ideal nanotopological space is denoted by \((U, \mathcal{N}, I)\). The nano-interior and nano-closure of a subset \( A \) of \( U \) are denoted by \( I^*(A) \) and \( C^*(A) \), respectively.

**Definition 1.1.** A subset \( A \) of a space \((U, \mathcal{N})\) is called

(i) nano \( \alpha\)-open if \( A \subseteq I_n(C_n(I_n(A))) \), [4];
(ii) nano semi-open if \( A \subseteq C_n(I_n(A)) \), [4];
(iii) nano pre-open if \( A \subseteq I_n(C_n(A)) \), [4];
(iv) nano \( b\)-open if \( A \subseteq I_n(C_n(A)) \cup C_n(I_n(A)) \), [5];
(v) nano \( \beta\)-open if \( A \subseteq C_n(I_n(C_n(A))) \), [12].

The complements of the above mentioned sets are called their respective closed sets.

**Definition 1.2.** [4] A subset \( A \) of a nano space \((U, \mathcal{N})\) is called nano regular-open (written in short as \( nr\)-open) \( A = I_n(C_n(A)) \).

The complement of \( nr\)-open set is said to be a \( nr\)-closed set.

**Definition 1.3.** [1] Let \( A \) be a subset of a space \((U, \mathcal{N})\) is nano \( \pi\)-open (written in short as \( n\pi\)-open) if the finite union of \( nr\)-open sets.

The complement of \( n\pi\)-open set is said to be a \( n\pi\)-closed set.

**Definition 1.4.** A subset \( A \) of a space \((U, \mathcal{N})\) is called

(i) nano \( g\)-closed (written in short as \( ng\)-closed) if \( C_n(A) \subseteq B \), whenever \( A \subseteq B \) and \( B \) is \( n\)-open, [2];
(ii) nano \( \pi g\)-closed (written in short as \( n\pi g\)-closed) if \( C_n(A) \subseteq B \), whenever \( A \subseteq B \) and \( B \) is \( n\pi\)-open, [9];
(iii) nano \( \alpha g\)-closed (written in short as \( n\alpha g\)-closed) if \( C_{n\alpha}(A) \subseteq B \), whenever \( A \subseteq B \) and \( B \) is \( n\alpha\)-open, [9];
(iv) nano \( \pi g\alpha\)-closed (written in short as \( n\pi g\alpha\)-closed) if \( C_{n\alpha}(A) \subseteq B \) whenever \( A \subseteq B \) and \( B \) is \( n\pi\)-open, [10].

The complements of the above mentioned sets are called their respective open sets.
Definition 1.5. [6] A subset $A$ of a space $(U, N, I)$ is $n^*$-dense in itself (resp. $n^*$-perfect and $n^*$-closed) if $A \subseteq A_n^*$ (resp. $A = A_n^{*o}$, $A_n^* \subseteq A$).

The complement of a $n^*$-closed set is said to be a $n^*$-open set.

Definition 1.6. [7] An ideal $I$ in a space $(U, N, I)$ is called $\mathfrak{K}$-codense ideal if $\mathfrak{K} \cap I = \{\phi\}$.

Definition 1.7. [11] A subset $A$ of space $(U, N, I)$ is said to be

(i) nano $\alpha$-open (written in short as $\alpha$-$nI$-open) if $A \subseteq I_n(C_n^*(I_n(A)))$,
(ii) nano semi-$\alpha$-open (written in short as semi-$\alpha$-$nI$-open) if $A \subseteq C_n^*(I_n(A))$,
(iii) nano pre-$\alpha$-open (written in short as pre-$\alpha$-$nI$-open) if $A \subseteq I_n(C_n^*(A))$,
(iv) nano $\beta$-$I$-open (written in short as $\beta$-$nI$-open) if $A \subseteq C_n^*(I_n(C_n^*(A)))$.

The complements of the above mentioned sets are called their respective closed sets.

Definition 1.8. A subset $A$ of a space $(U, N, I)$ is called a

(i) nano $I_g$-closed (written in short as $I_{ng}$-closed) if $A_n^* \subseteq B$ whenever $A \subseteq B$ and $B$ is $n$-open, [6];
(ii) nano $I_\omega$-closed (or) nano $I_g$-closed (written in short as $I_{ng\omega}$-closed) if $A_n^* \subseteq B$ whenever $A \subseteq B$ and $B$ is $n\pi$-open, [8].

The complements of the above mentioned sets are called their respective open sets.

2. $\pi g$-CLOSED SETS IN IDEAL NANOTOPOLOGICAL SPACES

Definition 2.1. A subset $A$ of an ideal nano space $(U, N, I)$ is called a nano $I_{ng}$-closed (written in short as $I_{ng\pi}$-closed) if $A \subseteq H$, $H \in n\pi$-open $\Rightarrow A_n^* \subseteq H$.

Nano $I_{ng}$-open (written in short as $I_{ng\pi}$-open) if $A = H - A$ (where $A$ denotes the complement operator and $A$ is $I_{ng\pi}$-closed).

Definition 2.2. A subset $A$ of an ideal nano space $(U, N, I)$ is called a

(i) nano $\mathcal{D}_I$-set if $A = H \cap V$, where $H$ is a $n\pi$-open set and $V$ is a $n^*$-perfect set.
(ii) nano $\mathcal{B}_I$-set if $A = H \cap V$, where $H$ is a $n\pi$-open set and $V$ is a $n^*$-closed set.
Theorem 2.1. Each $n\pi g$-closed set is $I_{n\pi g}$-closed.

Proof. Let $A$ be a every $n\pi g$-closed set. Then $A \subseteq H$, $H \in n\pi$-open $\implies C_n(A) \subseteq H$. Since $A^*_n \subseteq C_n(A) \subseteq H$, we have $A^*_n \subseteq H$ and hence $A$ is $I_{n\pi g}$-closed. \hfill \Box

Theorem 2.2. If $(U, \mathcal{N}, I)$ is any ideal nano space and $A \subseteq U$, then the following hold.

(i) If $I = \phi$, then $A$ is $I_{n\pi g}$-closed $\iff$ $A$ is $n\pi g$-closed.

(ii) If $I = \aleph_s$, then $A$ is $I_{n\pi g}$-closed $\iff$ $A$ is $n\pi g\alpha$-closed.

Proof. The proof follows from the fact that $A^*_n(\{\phi\}) = C_n(A)$ and $A^*_n(\aleph_s) = C_{n\alpha}(A)$. \hfill \Box

Theorem 2.3. If $A$ and $B$ is $I_{n\pi g}$-closed then $A \cup B$ is $I_{n\pi g}$-closed.

Proof. Suppose that $A \cup B \subseteq H$ and $H$ is $n\pi$-open, then $A, B \subseteq H$. Since $A$ and $B$ are $I_{n\pi g}$-closed, $A^*_n \subseteq H$ and $B^*_n \subseteq H$. Thus, $A \cup B$ is $I_{n\pi g}$-closed. \hfill \Box

Theorem 2.4. If a subset $A$ of $(U, \mathcal{N}, I)$ is $I_{n\pi g}$-closed, then $C^*_n(A) - A$ contains no nonempty $n\pi$-closed set.

Proof. Suppose that $A$ is $I_{n\pi g}$-closed and $F$ be a $n\pi$-closed subset of $C^*_n(A) - A$. Then $A \subseteq U - F$. Since $U - F$ is $n\pi$-open and $A$ is $I_{n\pi g}$-closed, $C^*_n(A) \subseteq U - F$.

Consequently, $F \subseteq U - C^*_n(A)$. We have $F \subseteq C^*_n(A)$. Thus, $F \subseteq C^*_n(A) \cap (U - C^*_n(A)) = \phi$ and so $C^*_n(A) - A$ contains no nonempty $n\pi$-closed set. \hfill \Box

Corollary 2.1. Let $(U, \mathcal{N}, I)$ be an ideal nano space and $A$ be an $I_{n\pi g}$-closed set. Then the following are equivalent.

(i) $A$ is a $n\ast$-closed set.

(ii) $C^*_n(A) - A$ is a $n\pi$-closed set.

(iii) $A^*_n - A$ is a $n\pi$-closed set.

Proof. (i) $\implies$ (ii) : If $A$ is $n\ast$-closed set, then $C^*_n(A) - A = \phi$ and so $C^*_n(A) - A$ is $n\pi$-closed.

(ii) $\implies$ (i) : Suppose $C^*_n(A) - A$ is $n\pi$-closed. Since $A$ is $I_{n\pi g}$-closed, By Theorem 2.4 $C^*_n(A) - A = \phi$ and so $A$ is $n\ast$-closed.

(ii) $\iff$ (iii) : Follows from the fact that $C^*_n(A) - A = A^*_n - A$. \hfill \Box

Theorem 2.5. In a space $(U, \mathcal{N}, I)$, every subset is $I_{n\pi g}$-closed $\iff$ every $n\pi$-open set is $n\ast$-closed.
Theorem 2.10. For a subset $n \star$ is a $H$ and $V = A \ I$

Proof. Suppose that $I H$ whenever $A$ is any $A$-closed set. Since $A$ is $n \star$-open and the only $n \star$-open set containing $\{x\}^c$ is the space $(U, N, I)$ itself.

Therefore, $C^*_n(\{x\})^c U$ and so $\{x\}^c$ is $I_{n^g}$-closed. $\square$

Remark 2.1. If $A$ is $n \star$-open and $I_{n^g}$-closed, then $A$ is $n \star$-closed.

Theorem 2.6. For each $x \in (U, N, I)$ either $\{x\}$ is $n \pi$-closed or $\{x\}^c$ is $I_{n^g}$-closed.

Proof. Suppose that $\{x\}$ is not $n \pi$-closed, then $\{x\}^c$ is not $n \pi$-open and the only $n \pi$-open set containing $\{x\}^c$ is the space $(U, N, I)$ itself.

Therefore, $C^*_n(\{x\}) \subseteq U$ and so $\{x\}^c$ is $I_{n^g}$-closed. $\square$

Theorem 2.7. If $A$ is an $I_{n^g}$-closed set such that $A \subseteq B \subseteq A^*$, then $B$ is also an $I_{n^g}$-closed set.

Proof. Let $H$ be any $n \pi$-open set such that $B \subseteq H$, then $A \subseteq H$. Since $A$ is $I_{n^g}$-closed, we have $A^*_n \subseteq H$. Now, $B^*_n \subseteq (A^*_n)^* \subseteq A^*_n \subseteq H$. Therefore, $B$ is $I_{n^g}$-closed. $\square$

Theorem 2.8. A subset $A$ of an ideal nano space $(U, N, I)$ is $I_{n^g}$-open $\iff F \subseteq I^*_n(A)$ whenever $F$ is $n \pi$-closed and $F \subseteq A$.

Proof. Suppose that $F \subseteq I^*_n(A)$ whenever $F$ is $n \pi$-closed and $F \subseteq A$. Let $A^c \subseteq H$, whenever $H$ is $n \pi$-open. Then $H^c \subseteq A$ and $H^c$ is $n \pi$-closed, therefore $H^c \subseteq I^*_n(A)$, which implies that $C^*_n(A^c) \subseteq H$. Hence, $A^c$ is $I_{n^g}$-closed and so $A$ is $I_{n^g}$-open. Conversely, suppose that $A$ is $I_{n^g}$-open, $F \subseteq A$ and $F$ is $n \pi$-closed. Then $F^c$ is $n \pi$-open and $A^c \subseteq F^c$. Therefore, $C^*_n(A^c) \subseteq F^c$ and so $F \subseteq I^*_n(A)$. $\square$

Theorem 2.9. A subset $A$ of an ideal nano space $(U, N, I)$ is a nano $\mathcal{D}$-set and an $I_{n^g}$-closed set, then $A$ is a $n \star$-closed set.

Proof. Let $A$ be a nano $\mathcal{D}$-set and a $I_{n^g}$-closed set. Since $A$ is a nano $\mathcal{D}$-set, $A = H \cap V$, where $H$ is a $n \pi$-open set and $V$ is a $n \star$-perfect set. Now, $A = H \cap V \subseteq H$ and $A$ is a $I_{n^g}$-closed set implies that $A^*_n \subseteq H$. Also, $A = H \cap V \subseteq V$ and $V$ is $n \star$-perfect set implies that $A^*_n \subseteq V$. Thus, $A^*_n \subseteq H \cap V = A$. Hence, $A$ is a $n \star$-closed set. $\square$

Theorem 2.10. For a subset $A$ of an ideal nano space $(U, N, I)$, $A$ is a $n \star$-closed set $\iff A$ is a nano $\mathcal{B}$-set and a $I_{n^g}$-closed set.
Definition 3.3. A map $f: (U, N, I) \rightarrow (F, \mathcal{X})$ is called nano $I_{ng}$-continuous (written in short as $I_{ng}$-continuous) if $f^{-1}(A)$ is $I_{ng}$-closed in $(U, N, I)$ for every $n$-closed set $A$ of $F$.

Definition 3.2. A map $f: (U, N) \rightarrow (F, \mathcal{X})$ is called a

(i) a nano $\pi$-space (written in short as $n\pi$-space) if $f(A)$ is $n\pi$-closed in $(F, \mathcal{X})$ for every $n\pi$-closed set $A$ in $(U, N)$.

(ii) a nano regular map (written in short as $nr$-map) if $f^{-1}(A)$ is $nr$-closed in $(U, N)$ for every $nr$-closed set $K$ of $F$.

Theorem 3.1. For a map $f: (U, N, I) \rightarrow (F, \mathcal{X})$, the following hold.

(i) $f$ is $n\pi g$-continuous $\Rightarrow$ $f$ is $I_{ng}$-continuous.

(ii) $f$ is $I_{ng}$-continuous $\Rightarrow$ $f$ is $I_{ng}$-continuous.

Definition 3.3. A map $f: (U, N, I) \rightarrow (F, \mathcal{X}, I)$ is called nano $I_{ng}$-irresolute (written in short as $I_{ng}$-irresolute) if $f^{-1}(A)$ is $I_{ng}$-closed in $(U, N, I)$ for every $I_{ng}$-closed set $A$ of $(F, \mathcal{X}, I)$.

Theorem 3.2. If $f: (U, N, I) \rightarrow (F, \mathcal{X}, I)$ is $I_{ng}$-continuous and $n\pi$-space, then $f$ is $I_{ng}$-irresolute.

Proof. Assume that $A$ is $I_{ng}$-closed in $F$. Let $f^{-1}(A) \subseteq H$, where $H$ is $n\pi$-open in $U$. Then $(U - H) \subseteq f^{-1}(F - A)$ and hence $f(U - H) \subseteq F - A$. Since $f$ is $n\pi$-space, $f(U - H)$ is $n\pi$-closed. Then, since $F - A$ is $I_{ng}$-open. By Theorem 2.8, $f(U - H) \subseteq I_n^*(F - A) = F - C_n^*(A)$. Thus, $f^{-1}(C_n^*(A)) \subseteq H$. Since $f$ is $I_{ng}$-continuous, $f^{-1}(C_n^*(A))$ is $I_{ng}$-closed. Therefore, $C_n^*(f^{-1}(C_n^*(A))) \subseteq H$.

3. On Nano $I_{ng}$-Continuous Maps
and hence $C_n(f^{-1}(A)) \subseteq C_n(f^{-1}(C_n(A))) \subseteq H$ which proves that $f^{-1}(A)$ is $I_{n\pi g}$-closed and therefore $f$ is $I_{n\pi g}$-irresolute.

\[\square\]

**Definition 3.4.** A map $f : (U, \mathcal{N}, I) \to (F, \mathcal{X})$ is called almost nano $I_{\pi g}$-continuous (written in sort as almost $I_{n\pi g}$-continuous) if $f^{-1}(A)$ is $I_{n\pi g}$-closed in $(U, \mathcal{N}, I)$ for every $A$ is $n$-regular closed in $F$.

**Theorem 3.3.** For a map $f : (U, \mathcal{N}, I) \to (F, \mathcal{X})$, the following are equivalent.

(i) $f$ is almost $I_{n\pi g}$-continuous.

(ii) $f^{-1}(A) \in I_{n\pi g}$-open for every $A$ is $n$-regular open in $F$.

(iii) $f^{-1}(I_n(C_n(A))) \in I_{n\pi g}$-open for every $A \in \mathcal{X}$.

(iv) $f^{-1}(C_n(I_n(A))) \in I_{n\pi g}$-closed for every $n$-closed set $A$ of $F$.

**Proof.** (i) $\iff$ (ii) : Obvious.

(ii) $\iff$ (iii) : Assuming that $A$ is $n$-regular open in $F$, we have $A = I_n(C_n(A))$ and $f^{-1}(I_n(C_n(A))) \in I_{n\pi g}$-open. Conversely, suppose $A \in \mathcal{X}$, we have $I_n(C_n(A)) \in n$-regular open ($F$) and $f^{-1}(I_n(C_n(A))) \in I_{n\pi g}$-open.

(iii) $\iff$ (iv) : Let $A$ be a $n$-closed set in $F$. Then $F - A \in \mathcal{X}$. We have $f^{-1}(I_n(C_n(F - A))) = f^{-1}(F - (C_n(I_n(A)))) = U - f^{-1}(C_n(I_n(A))) \in I_{n\pi g}$-open. Hence, $f^{-1}(I_n(C_n(A))) \in I_{n\pi g}$-closed. Converse can be obtained similarly. $\square$

**Theorem 3.4.** The following hold for the maps $f : (U, \mathcal{N}, I) \to (F, \mathcal{X}, J)$ and $g : (F, \mathcal{X}, J) \to (G, \mathcal{M})$,

(i) $g \circ f$ is $I_{n\pi g}$-continuous, if $f$ is almost $I_{n\pi g}$-continuous and $g$ is completely nano continuous.

(ii) $g \circ f$ is $I_{\pi g}$-continuous, if $f$ is $I_{n\pi g}$-continuous and $g$ is nano continuous.

(iii) $g \circ f$ is $I_{\pi g}$-continuous, if $f$ is $I_{n\pi g}$-irresolute and $g$ is $I_{n\pi g}$-continuous.

(iv) $g \circ f$ is almost $I_{\pi g}$-continuous, if $f$ is almost $I_{n\pi g}$-continuous and $g$ is nano $n$-map.

(v) $g \circ f$ is almost $I_{\pi g}$-continuous, if $f$ is $I_{n\pi g}$-irresolute and $g$ is almost $I_{n\pi g}$-continuous.

(vi) $g \circ f$ is almost $I_{\pi g}$-continuous, if $f$ is $I_{n\pi g}$-continuous and $g$ is almost $I_{n\pi g}$-continuous.

**Definition 3.5.** A map $f : (U, \mathcal{N}, I) \to (F, \mathcal{X})$ is called nano $\mathcal{B}_I$-continuous (written in sort as $\mathcal{B}_{nI}$-continuous) if $f^{-1}(A)$ is nano $\mathcal{B}_I$-set in $(U, \mathcal{N}, I)$ for every $n$-closed set $A$ of $F$. 
Theorem 3.5. A map \( f : (U, N, I) \to (F, \mathcal{X}) \) is \( n^* \)-continuous \( \iff \) \( \mathcal{B}_{nI} \)-continuous and \( I_{n\pi g} \)-continuous.

Proof. This is an immediate consequence of Theorem 2.10. \( \square \)

Remark 3.1. The concepts of \( \mathcal{B}_{nI} \)-continuity and the concepts of \( I_{n\pi g} \)-continuity are independent of each other as shown in the following Example.

Example 1. Let \( U = \{a, b, c\} \) be a non empty finite set with

(i) \( U/R = \{\{a\}, \{b\}, \{c\}\} \) and \( X = \{a, b\} \) then \( \mathcal{N} = \{\phi, U, \{a\}, \{b\}, \{a, b\}\} \).

(ii) \( U/R = \{\{a, b\}, \{c\}\} \) and \( X = \{b, c\} \) then \( \mathcal{X} = \{\phi, U, \{c\}, \{a, b\}\} \).

(iii) \( U/R = \{\{b, c\}, \{a\}\} \) and \( X = \{b, c\} \) then \( \mathcal{M} = \{\phi, U, \{b, c\}\} \).

And let ideal be \( I = \{\phi, \{c\}\} \).

In the ideal nano space \( (U, N, I) \), then

(i) the identity function \( F : (U, N, I) \to (U, \mathcal{M}) \) is \( \mathcal{B}_{nI} \)-continuous but not \( \mathcal{B}_{nI} \)-continuous.

(ii) the identity function \( G : (U, \mathcal{X}, I) \to (U, \mathcal{M}) \) is \( I_{n\pi g} \)-continuous but not \( \mathcal{B}_{nI} \)-continuous.

References


School of Mathematics, Madurai Kamaraj University,
Madurai-21, Tamil Nadu, India
E-mail address: jiometha@yahoo.com

Department of Mathematics
School of Mathematics, Madurai Kamaraj University,
Madurai-21, Tamil Nadu, India
E-mail address: rasoka_mku@yahoo.co.in

Department of Mathematics
Tirunelveli Dakshina Mara Nadar Sangam College, T. Kallikulam - 627 113
Tirunelveli District, Tamil Nadu, India
E-mail address: sekarmelakka@gmail.com