Oscillation Criteria for Second Order Advanced Type Nonlinear Neutral Difference Equations

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Abstract. The purpose of the indicating paper is to investigate the oscillation of the second (2nd) order Advanced type Neutral Difference Equation (NDE)
\[ \Delta(a_n(\Delta \chi_n + p_n \chi_\tau(n))) + q_n \chi_\sigma(n) = 0. \]
The obtained results are up to date, enhance and rounding off some of the existing results. Some model problems are imparted to explain the importance of the leading results.

1. Introduction

In the indicating paper, we shall investigate the oscillation criteria of the results of the difference equation of the advanced type

\[ \Delta(a_n(\Delta(Z_n))^{\alpha}) + q_n \chi_\sigma(n) = 0, \quad n \geq n_0 > 0 \]

where \( Z_n = \chi_n + p_n \chi_\tau(n) \), contingent to under the restrictions: \((H_1)\{a_n\}, \{q_n\}\) are positive (+ve) real sequences and \( \{p_n\} \) is a nonnegative real sequence for all \( n \geq n_0 \); \((H_2)\{\tau(n)\}\) and \( \{\sigma(n)\}\) are nondecreasing sequence of integers such that \( \sigma(n) \geq n + 1 \) and \( \tau(n) \to \infty \) as \( n \to \infty \); \((H_3)0 \leq p_n < \infty, \) and \( \tau \circ \infty = \sigma \circ \tau; \)
\((H_4)\alpha\) is a proportion of odd +ve integers.

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By a key result of the equation (1.1), we noted a nontrivial real sequence \( \{\chi_n\} \) which satisfies equation (1.1) for all \( n \geq n_0 \). We observed the certain results of equation (1.1) which exist for all \( n \geq N \geq n_0 \) and satisfy the condition \( \{|\chi_n| : n \geq N_1\} > 0 \) for any \( N_1 > N \). As usual, an answer \( \{\chi_n\} \) of equation (1.1) is named oscillatory if it’s neither eventually positive nor eventually negative and otherwise it’s called nonoscillatory. The equation itself is named oscillatory if all its solutions oscillate.

Advanced type difference equations can obtain applications in a number of problems in the world where evolvement rate calculated not only on the present but also on the future. Such type of equations arise in population dynamics and economic problems where such situations are thought to occur [1].

Oscillation solutions of NDE. Major conclusions clarify with the oscillation properties of delay difference equations of neutral type, see for example in [1,2] and the citations contained therein. However few results available for the oscillation of solutions of advanced type difference equations [3, 4, 6, 7]. Consequently here, we have to study the oscillatory behavior of second order NDE of the advanced type (1.1).

2. OSCILLATION RESULTS

For our further references, we denote and assume that

\[
R_n = \sum_{s=n_0}^{n-1} \frac{1}{s^\alpha} \rightarrow \infty \text{as} \ n \rightarrow \infty
\]

Let us consider the equation (1.1) and handling with the +ve solutions of equation (1.1) since the result for the opposite case is alike.

**Lemma 2.1.** If \( \{x_n\} \) be a +ve solution of equation (1.1). Then the relating sequence \( \{z_n\} \) satisfies

\[
Z_n > 0, a_n(\Delta Z_n)^\alpha > 0, \Delta(a_n(\Delta Z_n)^\alpha) > 0
\]

eventually. Moreover \( \{\frac{z_n}{R_n}\} \) is nonincreasing eventually.

**Proof.** Let \( \{x_n\} \) be a +ve solution of equation (1.1). Then it follows from (1.1) that

\[
\Delta(a_n(\Delta Z_n)^\alpha) = -q_n \chi_{\sigma(n)} < 0.
\]
Hence, $a_n(\Delta Z_n)^{\alpha}$ is decreasing and thus either $\Delta Z_n > 0$ or $\Delta Z_n < 0$, eventually. If we set $\Delta Z_n < 0$, then $a_n(\Delta Z_n)^{\alpha} < -M < 0$ and summing this inequality from $n_1$ to $n - 1$, we obtain

$$Z_n \leq Z_{n_1} - M \frac{1}{a_n^{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\alpha}} \to -\infty$$

as $n \to \infty$. This contradicts the positivity of $Z_n$ and therefore (2.1) holds. Since $a_n(\Delta Z_n)^{\alpha}$ is decreasing, we have

$$Z_n \geq a_n^{\frac{1}{\alpha}} \Delta Z_n \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\alpha}}$$

or

$$Z_n \geq R_n a_n^{\frac{1}{\alpha}} \Delta Z_n$$

and hence

$$\Delta \left( \frac{Z_n}{R_n} \right) = R_n a_n^{\frac{1}{\alpha}} \Delta Z_n - Z_n$$

that is, $\{ \frac{z_n}{R_n} \}$ is nonincreasing eventually. $\square$

**Lemma 2.2.** Let $x > 0, y > 0$ and $\alpha \in (0, \infty)$. Then

$$x^{\alpha} + y^{\alpha} \geq \frac{1}{2^{\alpha-1}} (x + y)^{\alpha} \text{ for } \alpha \geq 1,$$

and

$$x^{\alpha} + y^{\alpha} \geq (x + y)^{\alpha} \text{ for } 0 < \alpha < 1.$$

**Proof.** The constructed proof of the lemma can be done in [5]. For our further references, let us set

$$Q(n) = \min\{q_n, q_{r(n)}\}$$

$$Q^*(n) = \begin{cases} Q(n) & \text{if } \alpha \geq 1 \\ Q(n) & \text{if } 0 < \alpha < 1 \end{cases}$$

$$Q_1(n) = \frac{1}{a_n} \sum_{s=n}^{\infty} Q^*(s) \frac{1}{a_s^{\alpha}},$$

and

$$Q_2(n) = \left( Q^*(n) \sum_{s=n}^{\infty} \frac{1}{a_n^{\alpha}} \right)^{\frac{1}{\alpha}},$$

where $n \geq n_1, n_1$ is sufficiently large integer. $\square$
Theorem 2.1. Let $\tau(n) \geq n$. Assume that at least one of the first order advanced type difference inequalities

\begin{align}
\Delta W_n - \frac{1}{(1 + p)^\frac{1}{2}} Q_1(n) w_{\sigma(n)} &\geq 0, \\
\Delta W_n - \frac{1}{(1 + p)^\frac{1}{2}} Q_2(n) w_{\sigma(n)} &\geq 0,
\end{align}

has no +ve solution. Then each result of the equation (1.1) is oscillatory.

Proof. Let us consider $\{x_n\}$ be a +ve solution of equation (1.1), and also the companion sequence $\{Z_n\}$ satisfies

\begin{equation}
Z_{\sigma(n)} = \chi_{\sigma(n)} + p\sigma(n)\chi_{\tau(\sigma(n))} \leq \chi_{\sigma(n)} + p\chi_{\tau(n)}.
\end{equation}

Here we have used the assumption $(H_3)$. From equation (1.1), we have

\begin{align}
0 &= \Delta(a_{\tau(n)}(\Delta Z_{\tau(n)})^\alpha) + q_{\tau(n)}^{\alpha} \chi_{\sigma(\tau(n))} \\
&\geq p^\alpha \Delta(a_{\tau(n)}(\Delta Z_{\tau(n)})^\alpha) + q_{\tau(n)}^{\alpha} \chi_{\sigma(\tau(n))} + p^\alpha q_n \chi_{\sigma(\tau(n))} \leq 0
\end{align}

Combining (1.1) and (2.5) yields

\begin{equation}
\Delta(a_n(\Delta(Z_n))^\alpha) + p^\alpha \Delta(a_{\tau(n)}(\Delta Z_{\tau(n)})^\alpha) + q_n \chi_{\sigma(\tau(n))} + p^\alpha q_n \chi_{\sigma(\tau(n))} \leq 0
\end{equation}

which in view of (2.4) and Lemma 2.2 gives

\begin{equation}
\Delta(a_n(\Delta(Z_n))^\alpha) + p^\alpha \Delta(a_{\tau(n)}(\Delta Z_{\tau(n)})^\alpha) + Q^*(n) Z_{\sigma(n)}^\alpha.
\end{equation}

Summing (2.6) from $n \to \infty$, one obtains

\begin{equation}
a_n(\Delta Z_n)^\alpha + p^\alpha a_{\tau(n)}(\Delta Z_{\tau(n)})^\alpha \geq \sum_{s=n}^{\infty} Q^*(s) Z_{\sigma(s)}^\alpha.
\end{equation}

On the other hand, since $a_n(\Delta(Z_n))^\alpha$ is decreasing and $\tau(n) \geq n$, it follows from (2.7) that

\begin{equation}
a_n(\Delta Z_n)^\alpha(1 + p^\alpha) \geq \sum_{s=n}^{\infty} Q^*(s) Z_{\sigma(s)}^\alpha
\end{equation}

or

\begin{equation}
\Delta Z_n \geq \frac{1}{(1 + p^\alpha)^\frac{1}{2}} \left( \frac{1}{a_n} \sum_{s=n}^{\infty} Q^*(s) Z_{\sigma(s)}^\alpha \right)^{\frac{1}{2}}.
\end{equation}
Applying $Z_{\sigma(n)}$ is increasing, a summation from $n_1$ to $n - 1$, yields

$$Z_n \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Z_{\sigma(s)} \left( \frac{1}{a_s} \sum_{t=s}^{\infty} Q^*(t) Z_{\sigma(t)}^\alpha \right)^{\frac{1}{\alpha}}.$$ 

That is,

(2.9) $$Z_n \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Q_1(s) Z_{\sigma(s)}.$$ 

Let us assume that the right side of (2.9) by $W_n$. Then $W_n > 0$ and using $Z_n \geq W_n$, we have

$$\Delta W_n = \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} Q_1(n) Z_{\sigma(n)} \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} Q_1(n) W_{\sigma(n)}.$$ 

Thus $\{W_n\}$ is a +ve solution of (2.2), this is a inconsistency. Hence the absence of +ve solution of (2.2) gives the oscillation of equation (1.1).

Next, we have to prove that the absence of the +ve solutions of (2.3) also gives the oscillation criteria of equation (1.1). Summation (2.8) from $n_1$ to $n - 1$, gives

$$Z_n \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} \left( \frac{1}{a_s} \sum_{t=s}^{\infty} Q^*(t) Z_{\sigma(t)}^\alpha \right)^{\frac{1}{\alpha}}.$$ 

$$\geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} \left( \frac{1}{a_s} \sum_{t=s}^{n-1} Q^*(t) Z_{\sigma(t)}^\alpha \right)^{\frac{1}{\alpha}}.$$ 

$$= \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Z_{\sigma(s)} \left( Q^*(s) \sum_{t=s}^{n-1} \frac{1}{a_t} \right)^{\frac{1}{\alpha}}.$$ 

That is,

(2.10) $$Z_n \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Q_2(s) Z_{\sigma(s)}.$$ 

Let $\{W_n\}$ be the right side of (2.10). Then $W_n > 0$ and using $Z_n \geq W_n$, one can see that $\{W_n\}$ is a +ve solution of the inequity (2.3). This counterstatement gives the completed proof of the theorem. □
Corollary 2.1. Let \( \tau(n) \geq n \) and \( \sigma(n) = n + k \) where \( k \geq 2 \) is an integer. If consider that at least one of the following conditions

\[
\lim_{n \to \infty} \inf \sum_{s=n+1}^{n+k-1} Q_1(s) > (1 + p^\alpha)^\frac{1}{\alpha} \left( \frac{k - 1}{k} \right)^k,
\]

(2.11)

\[
\lim_{n \to \infty} \inf \sum_{s=n+1}^{n+k-1} Q_2(s) > (1 + p^\alpha)^\frac{1}{\alpha} \left( \frac{k - 1}{k} \right)^k,
\]

(2.12)

holds. Then every solution of equation (1.1) oscillates.

Proof. From Lemma 6.1.7 of [2], one can see that (2.2) and (2.3) has positive solutions provided that condition (2.11) and (2.12) hold respectively. The conclusion now follows from Theorem 2.1. For our next results, let us define

\[
Q_3(n) = \left( \frac{1}{a_{\tau(n)}} \sum_{s=n}^{\infty} Q^*(s) \right)^\frac{1}{\alpha}, Q_4(n) = \left( Q^*(n) \sum_{s=n}^{\infty} \frac{1}{a_{\tau(n)}} \right)^\frac{1}{\alpha}.
\]

□

Theorem 2.2. Let \( \tau(n) \leq n \). Assume that at least one of the first order advanced difference inequalities

\[
\Delta W_n - \frac{1}{(1 + p^\alpha)^\frac{1}{\alpha}} Q_3(n) W_{\tau^{-1}(\sigma(n))} \geq 0,
\]

(2.13)

\[
\Delta W_n - \frac{1}{(1 + p^\alpha)^\frac{1}{\alpha}} Q_4(n) W_{\tau^{-1}(\sigma(n))} \geq 0,
\]

(2.14)

has no positive solution. Then each result of the equation (1.1) is oscillatory.

Proof. Let us consider \( \{x_n\} \) be a +ve solution of equation (1.1), and also from the proof of Theorem 2.1, the \( \{Z_n\} \) sequence satisfies (2.7). Since \( a_n (\Delta Z_n)^{\alpha} \) is decreasing and \( \tau(n) \leq n \), then it follows from (2.7) that

\[
a_{\tau(n)} (\Delta Z_{\tau(n)})^{\alpha}(1 + p^\alpha) \geq \sum_{s=n}^{\infty} Q^*(s) Z^\alpha_{\sigma(s)}
\]

(2.15)

or

\[
\Delta Z_{\tau(n)} \geq \frac{1}{(1 + p^\alpha)^\frac{1}{\alpha}} \left( \frac{1}{a_{\tau(n)}} \sum_{s=n}^{\infty} Q^*(s) Z^\alpha_{\sigma(s)} \right)^\frac{1}{\alpha}.
\]
Aggregating the last inequality from \( n_1 \) to \( n - 1 \), we obtain

\[
\Delta Z_{\tau(n)} \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Z_{\sigma(s)} \left( \frac{1}{a_{\tau(n)}} \sum_{s=n}^{\infty} Q^*(s) \right)^{\frac{1}{\alpha}}. 
\]

That is, (2.16)

\[
Z_{\tau(n)} \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Q_3(s) Z_{\sigma(s)}. 
\]

Let us denote the right side of (2.16) by \( \{ W_n \} \). Then \( W_n > 0 \) and \( Z_{\tau(n)} \geq W_n \) and one can see that \( \{ W_n \} \) is a \(+ve\) solution of the inequity (2.13). This gives refuse to accepts our assumption and hence the absence of the positive solutions of (2.13) signify the oscillation of equation (1.1).

Now, we shall prove that the absence of the \(+ve\) solutions of (2.14) signifies the oscillation of equation (1.1). Summing (2.15) from \( n_1 \) to \( n - 1 \), yields

\[
Z_{\tau(n)} \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} a_{\tau(s)} \left( \sum_{t=s}^{\infty} Q^*(t) Z_{\sigma(t)}^\alpha \right)^{\frac{1}{\alpha}} 
\]

\[
\geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} \left( \sum_{t=s}^{\infty} Q^*(t) Z_{\sigma(t)}^\alpha \right)^{\frac{1}{\alpha}} 
\]

\[
\geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Z_{\sigma(s)} \left( \sum_{t=n_1}^{n-1} \frac{1}{a_{\tau(t)}} \right)^{\frac{1}{\alpha}} 
\]

That is, (2.17)

\[
Z_{\tau(n)} \geq \frac{1}{(1 + p^\alpha)^{\frac{1}{\alpha}}} \sum_{s=n_1}^{n-1} Q_4(s) Z_{\sigma(s)}. 
\]

Let us define the right side of (2.17) by \( W_n \). Thus \( W_n > 0 \) and using \( Z_{\tau(n)} \geq W_n \), one can see that \( \{ W - n \} \) is a \(+ve\) solution of the inequity (2.14). This is a inconsistency and the proof of the theorem is completed.

\[\square\]

**Corollary 2.2.** Let \( \tau(n) = n - m \) and \( \sigma(n) = n + k \) where \( m \) is a nonnegative integer and \( k \geq 2 \) is a positive integer. If consider that at least one of the following conditions

\[
\lim_{n \to \infty} \inf_{s=n+1}^{n+m+k-1} Q_3(s) > \left( 1 + p^\alpha \right)^{\frac{k}{\alpha}} \left( \frac{m+k-1}{m+k} \right)^{m+k},
\]


\begin{equation}
\lim_{n \to \infty} \inf_{s=n+1}^{n+m+k-1} Q_4(s) > (1 + p^\alpha)\frac{1}{2} \left( \frac{m + k - 1}{m + k} \right)^{m+k},
\end{equation}
holds. Then every solution of equation (1.1) oscillates.

**Proof.** From Lemma 6.1.7 of [2], one can see that the inequalities (2.13) and (2.14) have no +ve solution provided that conditions (2.18) and (2.19) hold, respectively. The ending is now follows from Theorem 2.2.

\[\square\]

In the following we obtain a oscillation criteria using Riccati type transformation.

**Theorem 2.3.** Let \( \tau(n) > n \). Assume that there exists a nondecreasing +ve sequence \( \{\rho_n\} \) such that
\begin{equation}
\lim_{n \to \infty} \sup_{s=n+1}^{n} \left[ \rho_s Q^*(s) - \frac{(1 + p^\alpha) a_s \rho_s^{\alpha+1}}{(\alpha + 1)^{\alpha+1}} \rho_s^\alpha \right] = \infty.
\end{equation}
Then each result of the equation (1.1) is oscillatory.

**Proof.** Let us consider \( \{x_n\} \) be a +ve solution of equation (1.1), and also as in the proof of Theorem 2.1, the sequence \( \{Z_n\} \) satisfies (2.6). Define
\[\nu_n = \frac{\rho_n [a_n (\Delta Z_n)^\alpha + p^\alpha a_{\tau(n)} (\Delta Z_{\tau(n)})^\alpha]}{Z_n^\alpha}, n \geq n_1.\]
Then \( \nu_n > 0 \), and using \( a_n (\Delta Z_n)^\alpha \) is decreasing, we obtain
\begin{equation}
\Delta \nu_n = \Delta \rho_n \nu_{n+1} + \frac{\rho_n}{Z_{n+1}^\alpha} \Delta \left( a_n (\Delta Z_n)^\alpha + p^\alpha a_{\tau(n)} (\Delta Z_{\tau(n)})^\alpha \right) - \frac{\rho_n}{Z_{n+1}^\alpha} \nu_{n+1} \Delta Z_{n+1}^\alpha.
\end{equation}

By Mean Value Theorem
\[\Delta Z_n^\alpha \geq \alpha \frac{Z_{n+1}^\alpha}{Z_n} \Delta Z_n.\]
Using (2.13) and (2.6) in (2.21) one obtains
\begin{equation}
\Delta \nu_n \geq -\rho_n Q^*(n) + \frac{\Delta \rho_n}{\rho_{n+1}} \nu_{n+1} - \frac{\rho_n}{\rho_{n+1}} \nu_{n+1} \Delta \frac{Z_n}{Z_{n+1}}.
\end{equation}
Here we have use $Z_n^\alpha \geq Z_{n+1}^\alpha$. On the other hand from (2.6), and again using $a_n(\Delta Z_n)^\alpha$ is decreasing and $\tau(n) \geq n$, we have

$$\frac{\nu_{n+1}^\frac{1}{\alpha}}{\rho_{n+1}^\frac{1}{\alpha}(1 + p^{\alpha})^\frac{1}{\alpha}} \leq \frac{\Delta Z_{n+1}^\alpha}{Z_{n+1}^\alpha}.$$ 

Using the above inequality in (2.22), yields

$$\Delta \nu_n \geq -\rho_n Q^\ast(n) + \frac{\Delta \rho_n}{\rho_{n+1}} \nu_{n+1} - \frac{\alpha \rho_n^{1 + \frac{1}{\alpha}}}{\rho_{n+1}^{1 + \frac{1}{\alpha}} a_n^{\frac{1}{\alpha}} (1 + p^{\alpha})^\frac{1}{\alpha}},$$

where we have used $a_n^{\frac{1}{\alpha}}$ is decreasing. Now let $A = \frac{\Delta \rho_n}{\rho_{n+1}}$, $B = \frac{\rho_{n+1}^{1 + \frac{1}{\alpha}}}{\rho_{n+1}^{1 + \frac{1}{\alpha}} a_n^{\frac{1}{\alpha}} (1 + p^{\alpha})^\frac{1}{\alpha}}$ and in view of the inequality $Au - Bu^{1 + \frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha + 1)^\alpha + 1} A^{n+1} B^{-\alpha}$, we obtain from (2.23) that

$$\Delta \nu_n \leq -\rho_n Q^\ast(n) + \frac{(1 + p^{\alpha}) a_n}{(\alpha + 1)^{\alpha+1} \rho_n^{\alpha}} (\Delta \rho_n)^{\alpha+1}, n \geq n_1.$$ 

Summing the inequality from to, one obtains

$$\sum_{s=n_1}^{n} \left[ \rho_s Q^\ast(s) - \frac{(1 + p^{\alpha}) a_n}{(\alpha + 1)^{\alpha+1} \rho_n^{\alpha}} (\Delta \rho_n)^{\alpha+1} \right] \leq \nu_{n_1},$$

which contradicts (2.20) and the proof is now complete. □

**Theorem 2.4.** Let assume there exists a $+ve$ nondecreasing sequence such that

$$\lim_{n \to \infty} \sup \sum_{s=n_1}^{n} \left[ \rho_s Q^\ast(s) - \frac{(1 + p^{\alpha}) a_n}{(\alpha + 1)^{\alpha+1} \rho_n^{\alpha}} (\Delta \rho_n)^{\alpha+1} \right] = \infty.$$ 

Then each result of the equation (1.1) is oscillatory.

**Proof.** Let us consider $\{\chi_n\}$ be a $+ve$ solution of equation (1.1), and also from Theorem 2.1, the sequence $\{Z_n\}$ satisfies (2.6). Define

$$\nu_n = \rho_n \frac{a_n (\Delta Z_n)^\alpha + p^{\alpha} a_{\tau(n)} (\Delta Z_{\tau(n)})^\alpha}{Z_{\tau(n)}^\alpha}, n \geq n_1.$$ 

Then $\nu_n > 0$, and proceeding as in the proof of Theorem 2.5, we have

$$\Delta \nu_n \geq -\rho_n Q^\ast(n) + \frac{\Delta \rho_n}{\rho_{n+1}} \nu_{n+1} - \frac{\rho_n}{\rho_{n+1}} \nu_{n+1} \frac{\Delta Z_{\tau(n)}}{Z_{\tau(n)+1}}.$$
On the other hand by using $a_n(\Delta Z_n)^\alpha$ decreasing, we have from (2.25) that
\[
\frac{\nu_{n+1}^{\frac{1}{\alpha}}}{\rho_{n+1}^{\frac{1}{\alpha}}(1+p^n)^{\frac{1}{\alpha}}} \leq \frac{a_{\tau(n+1)}^{\frac{\alpha}{2}}\Delta Z_{\tau(n+1)}}{Z_{\tau(n+1)}}.
\]
Using the above inequality in (2.26), yields
\[
\Delta \nu_n \geq -\rho_n Q^*(n) + \frac{\Delta \rho_n}{\rho_{n+1}} \nu_{n+1} - \frac{\alpha \rho_n \nu_{n+1}^{1+\frac{1}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}} \sigma_{\tau(n)}(1+p^n)^{\frac{1}{\alpha}}}.
\]
The resting part of the proof is same as that of Theorem 2.5 and hence the details are neglected. The proof is completed.

\[\square\]

3. Examples

In this area, some examples to clarify the importance of the major solutions.

**Example 1.** Consider the second order advanced type neutral difference equation
\[\Delta \left(\frac{1}{n^2}(\Delta(\chi_n + 3\chi_{n+2}))^3\right) + \frac{\lambda}{n(n+1)} \chi_n^3 = 0, n \geq 1, \lambda > 0.\]
Here $a_n = \frac{1}{n^2}$, $p_n = 3$, $q_n = \frac{\lambda}{n(n+1)}$, $\tau(n) = n + 2$, $\sigma(n) = n + 3$ and $\alpha = 3$.
Then $Q(n) = \frac{\lambda}{n+2}$, $Q^*(n) = \frac{\lambda}{4(n+2)(n+3)}$ and $Q_1(n) = (\frac{\lambda}{4})^\frac{1}{2} \frac{n}{(n+2)^{\frac{1}{2}}}$. A simple computations shows that $R_n \to \infty$ as $n \to \infty$ and the condition (2.11) becomes
\[\lim_{n \to \infty} \inf_{n+1} \sum_{s=n+1}^{n+2} \left(\frac{\lambda}{4}\right)^\frac{1}{2} \frac{s}{(s+2)^{\frac{1}{2}}} = \lim_{n \to \infty} \inf_{n+1} \left(\frac{\lambda}{4}\right)^\frac{1}{2} \left[\frac{n+1}{(n+3)^{\frac{1}{2}}} + \frac{n+2}{(n+4)^{\frac{1}{2}}}\right] = \infty.
\]
Hence condition (2.11) is satisfied. Hence by Corollary 2.2, each result of equation (3.1) is oscillatory.

**Example 2.** Consider the 2nd order NDE
\[\Delta^2(\chi_n + p\chi_{n-2}) + \frac{\lambda}{n} \chi_{n+4} = 0, n \geq 1, \lambda > 0.\]
Here $a_n = 1$, $p_n = p > 0$, $q_n = \frac{\lambda}{n}$, $\tau(n) = n - 2$, $\sigma(n) = n + 4$ and $\alpha = 1$. Then $Q(n) = \frac{\lambda}{n}$, $Q^*(n) = \frac{\lambda}{n}$ and $Q_4(n) = \frac{\lambda}{n}(n-1)$. Now the condition (2.19) becomes
\[\lim_{n \to \infty} \inf_{n+1} \sum_{s=n+1}^{n+5} \frac{\lambda}{s} (s-1) = 5\lambda > (1+p) \left(\frac{5}{6}\right)^6.\]
Using Corollary 2.4, every result of equation (3.2) is oscillatory.

Then \( \lambda > (1 + p) \left( \frac{5}{6} \right)^6 \).

**Example 3.** Consider the 2nd order NDE

\[
(3.3) \quad \Delta \left( n(\Delta (x_n + p\chi_{2n}))^3 \right) + \frac{\lambda}{n} x_{3n}^3 = 0, \quad n \geq 1, \quad \lambda > 0.
\]

Here \( a_n = n, p_n = p > 0, q_n = \frac{\lambda}{n}, \tau(n) = 2n, \sigma(n) = 3n \) and \( \alpha = 3 \). Then \( Q(n) = \frac{\lambda}{2n}, Q^*(n) = \frac{\lambda}{5n} \) and by choosing \( \rho_n = 1 \), we see that condition (2.20) is satisfied. Therefore by Theorem 2.5, each result of (3.3) is oscillatory.

4. Conclusion

In this paper, we achieve a few new results for the oscillation of all solutions of the equation (1.1). The established results are enhanced and complement to the existing results for the neutral delay differential equations. It would be curiosity to achieve results similar to this paper without the restriction \( \tau \circ \sigma = \sigma \circ \tau \).

**References**


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