SOME CHARACTERIZATIONS OF PROPER POWER GRAPHS OF CERTAIN FINITE GROUPS

H. UMA¹ AND K. MANILAL

ABSTRACT. Let $G$ be a finite group. The power graph $\mathcal{P}(G)$ is an undirected graph with vertex set $G$ where two distinct vertices $u$ and $v$ in $\mathcal{P}(G)$ are adjacent if $u = v^\alpha$ or $v = u^\beta$ for some $\alpha, \beta$ in $\mathbb{N}$. The proper power graph denoted by $\mathcal{P}^*(G)$ of the group $G$ is the graph obtained by deleting the identity element from $G$, i.e., $\mathcal{P}^*(G) = \mathcal{P}(G/e)$. We characterize the proper power graphs of certain finite groups such as $\mathbb{Z}_{p^n}$ (the cyclic group of order $p^n$), $\mathbb{Z}_{pq}$ (the cyclic group of order $p^2q$), $D_{p^n}$ (dihedral group of order $2p^n$), $D_{pq}$ (dihedral group of order $2pq$), $S_3$, $S_4$, $S_5$, $A_4$, $A_5$. We also examine various graphical parameters of the above graphs such as completeness, independence number, covering number etc. We also examine the necessary and sufficient condition for the completeness of a proper power graph.

1. INTRODUCTION

According to literature, several graphs have been constructed from various algebraic structures [2]. One such graph is the power graph. Chakrabarty et al. [1] has defined the power graph for a semigroup $S$ as a graph with vertex set $S$ where two distinct vertices $u$ and $v$ in $S$ are adjacent if $u = v^\alpha$ or $v = u^\beta$ for some $\alpha, \beta$ in $\mathbb{N}$. The proper power graph [4] is a subgraph of the power graph obtained by deleting the vertex corresponding to the identity element from the

¹corresponding author

2010 Mathematics Subject Classification. 05C25, 05C60, 05C69, 05C70.

Key words and phrases. Power graph, Proper Power graph, Cyclic groups.
power graph. The power graph and the proper power graph associated with a
group $G$ is denoted by $\mathcal{P}(G)$ and $\mathcal{P}^*(G)$ respectively.

2. Preliminaries and Basic Results

Throughout the entire text, we refer to $\mathbb{Z}$ and $\mathbb{N}$ as the set of all integers and
set of all positive integers, respectively. $\phi(n)$ denotes the Euler totient function
of an integer $n$.

A group $(G, \ast)$ [6] is a non-empty set with an associative binary operation
$\ast : G \times G \rightarrow G$ such that $G$ has a unique identity element $e$ and every element
in $G$ has a unique inverse. The cardinality of $G$, denoted by $o(G)$, is called the
order [6] of $G$. A group $G$ is said to be cyclic [6] if there exists an element
$a \in G$ such that $a^n = e$ for some $n \in \mathbb{N}$ (where $a^n$ represents $a \ast a \ast \ldots \ast a$.
A subgroup [6] $H$ of $G$ is a subset of $G$ which itself is a group with the same
identity and with respect to the same binary operation.

A permutation [6] of a set $S$ is a one-to-one onto function on $S$. Then $S_n$
denotes the symmetric group [6] of all permutations on $n$ symbols and $A_n$
denotes the alternating [6] group of all even permutations on $n$ symbols. Then
$A_n$ is a subgroup of $S_n$ with order $n! / 2$.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. Then the union [7]
of $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \cup \Gamma_2$ is the graph with vertex set $V_1 \cup V_2$ and edge
set $E_2 \cup E_2$. The join [7] of $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \vee \Gamma_2$ is the graph with
vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ together with all the edges joining all the
vertices in $V_1$ to all the vertices in $V_2$. For any connected graph $\Gamma$, $n\Gamma$ represents
a graph with $n$ components each isomorphic to $\Gamma$. Isomorphism between two
graphs (groups) $G_1$ and $G_2$ is denoted by $G_1 \cong G_2$. Let $G$ be a graph. Then $\Gamma(G)$
denote the automorphism group of $G$.

A subset $G$ of the vertex set $V$ of a graph $G$ is called independent [8] if no
two vertices of $S$ are adjacent in $G$. $S \subseteq V$ is a maximum independent [8] set of $G$ if $G$ has no independent set $S'$ with $|S'| > |S|$. A subset $K$ of $V$
is called a covering [8] of $G$ if every edge of $G$ is incident with atleast one
vertex of $K$. A covering $K$ is minimum if there is no covering $K'$ of $G$ such
that $|K'| > |K|$. The number of vertices in a maximum independent set of $G$ is
called the independence number [8] of $G$ and is denoted by $\alpha(G)$. The number
of vertices in a minimum covering of $G$ is the covering number $[8]$ of $G$ and is denoted by $\beta(G)$. A matching $[8]$ in $G$ is a set of independent edges. A matching $M$ of $G$ is maximum $[8]$ if $G$ has no matching $M'$ with $|M'| > |M|$. An edge covering $[8]$ of $G$ is a subset $L$ of $E$ such that every vertex of $G$ is incident to some edge of $L$. $\alpha'(G)$ is the cardinality of a maximum matching and $\beta'(G)$ is the size of a minimum edge covering of $G$ $[8]$.

The results enlisted below are useful for subsequent reading:

Result 2.1. $[6]$ Fundamental theorem of cyclic groups: - For each positive divisor $k$ of $n$, the set $\langle \frac{n}{k} \rangle$ is the unique subgroup of $\mathbb{Z}_n$ of order $k$.

Result 2.2. $[6]$ If $d$ is a positive divisor of $n$, the number of elements of order $d$ in a cyclic group of order $n$ is $\phi(n)$.

Result 2.3. $[6]$ Sylow’s first theorem: - Let $G$ be a finite group and let $p$ be a prime. If $p^k$ divides $|G|$, then $G$ has atleast one subgroup of order $p^k$.

Result 2.4. $[6]$ If $G$ is a group of order $pq$, where $p$ and $q$ are primes, then $G$ is either cyclic and isomorphic to $\mathbb{Z}_{pq}$ or $G$ is non-abelian.

Result 2.5. $[4]$ $\mathcal{P}(G) \cong K_1 \vee \mathcal{P}^*(G)$.

Result 2.6. $[8]$ For any graph $G$, $\alpha + \beta = n$.

Result 2.7. $[8]$ For any graph $G$, $\alpha' + \beta' = n$.

3. Completeness of proper power graphs

In this section, we derive some important results which determine the completeness of the proper power graphs of certain finite groups.

Theorem 3.1. For any prime $p$ and $n \geq 1$, $\mathcal{P}^*(\mathbb{Z}_{p^n})$ is a complete graph on $p^n - 1$ vertices. In other words,

$$\mathcal{P}^*(\mathbb{Z}_{p^n}) \cong K_{p^n-1}.$$ 

Proof. For any $x, y \in \mathbb{Z}_{p^n}/\{0\}$, either $x^{p^n} = y^{p^n} = 0 \implies o(x)|p^n$ and $o(y)|p^n$. Thus $o(x) = p^{n_1}$ and $o(y) = p^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$ such that $n_1, n_2 < n$. Then either $n_1 \leq n_2$ or $n_1 \geq n_2$. In both the cases, either $x \in \langle y \rangle$ or $y \in \langle x \rangle$. That is, either $x^m = y$ or $y = x^k$ for some $m, k \in \mathbb{N}$. Thus, all non-zero elements in $\mathbb{Z}_{p^n}$ are adjacent to each other. Thus, $\mathcal{P}^*(\mathbb{Z}_{p^n})$ is complete with order $p^n - 1$. Hence the result. $\square$
Theorem 3.2. For any two distinct primes $p$ and $q$, $\mathcal{P}^*(\mathbb{Z}_{pq})$ is not complete. Moreover,

$$\mathcal{P}^*(\mathbb{Z}_{pq}) \cong (K_{p-1} \cup K_{q-1}) \lor K_{\phi(pq)}.$$ 

Proof. In order to complete the proof, we consider both the cases where both the primes are odd and one of them is even.

Assume that both are odd. By Sylow’s first theorem, $\langle p \rangle$ and $\langle q \rangle$ are subgroups of $\mathbb{Z}_{pq}$ and $\langle p \rangle \cap \langle q \rangle = \{0\}$. Thus, $p \notin \langle q \rangle$ and $q \notin \langle p \rangle$. Hence the vertices $p$ and $q$ are not adjacent. Hence the result. Now for the characterization part, from [3], we know that if $p$ and $q$ are distinct odd primes, $\mathcal{P}(\mathbb{Z}_{pq}) \cong (K_{p-1} \cup K_{q-1}) \lor K_{\phi(pq)}$. Since Result 2.5 holds, proof follows immediately.

Assume that one of them is even. Without loss of generality, assume $p$ is even i.e $p = 2$ and $q$ is odd. The vertex pairs $(2.1, p), (2.2, p), \ldots, (2.n, p)$ are not adjacent where $n$ is even and $n < p$. As in Theorem 3.1, let us examine whether $2.k \in \langle p \rangle$ or $p \in \langle 2.k \rangle$ where $k$ is even, $k < p$. Now, $\langle 2.k \rangle = \langle 2 \rangle$ and so $p \notin \langle 2.k \rangle$. Now, $\langle p \rangle = \{p, 1, p.2, \ldots, \}$. Thus $2.k \notin \langle p \rangle$. For the characterization part, by Fundamental theorem of cyclic groups, $\mathbb{Z}_{2p}$ contains exactly one subgroup of order 2 (isomorphic to $\mathbb{Z}_2$), order $p$ (isomorphic to $\mathbb{Z}_p$) and order $2p$. Hence $\mathcal{P}^*(\mathbb{Z}_{2p}) \cong (K_1 \cup K_{p-1}) \lor K_{p-1}$.

Hence combining both the cases, we get the desired result. □

Theorem 3.3. $\mathcal{P}^*(S_3) \cong 3K_1 \cup K_2$

Proof. We know that from [5], $\mathcal{P}(S_3) \cong [3K_1 \cup K_2] \lor K_1$. Then by Result 2.5, $\mathcal{P}^*(S_3) \lor K_1 \cong \mathcal{P}(S_3)$. Hence the result. □

Corollary 3.1. There is no group $G$ whose proper power graph is isomorphic to $K_5$.

Proof. $K_5$ has 5 vertices and inorder for a proper power graph to be isomorphic to $K_5$, then the group must be of order 6. Then by Result 2.4, the only two groups of order 6 are $\mathbb{Z}_6$ and $S_3$. But by theorem 3.2 and theorem 3.3, proper power graphs of $\mathbb{Z}_6$ and $S_3$ are not isomorphic to $K_5$. □

Theorem 3.4. For any two distinct primes $p$ and $q$, $\mathcal{P}^*(\mathbb{Z}_{p^2q})$ is not complete. Moreover,

$$\mathcal{P}^*(\mathbb{Z}_{p^2q}) \cong \left[\left[\left[K_{p-1} \lor K_{\phi(p^2)}\right] \lor K_{q-1}\right] \lor K_{\phi(pq)}\right] \lor K_{\phi(p^2q)}.$$
Proof. By Fundamental theorem of cyclic groups, there exists subgroups of order \( p, q \) and \( pq \). Let \( x, y \in \mathbb{Z}_{pq} \) such that \( o(x) = p \) and \( o(y) = q \). Then \( x \) and \( y \) are not adjacent (as in Theorem 3.2). Hence the result. Now to prove the characterization part, observe that \( \mathbb{Z}_{pq} \) contains elements of order \( p, p^2, q, pq, p^2q \). Now we observe that:

1. Every element of order \( p^2 \) is adjacent to every element of order \( p \), but not adjacent to an element of order \( q \).
2. Every element of order \( pq \) is adjacent to all elements of order \( p, p^2, q \).
3. Every element of order \( p^2q \) is adjacent to all elements of order \( p, p^2, q, pq \).

Hence the result.

\[ \square \]

Theorem 3.5. For any prime \( p \) and \( n \geq 1 \), \( \mathcal{P}^*(D_{p^n}) \cong [p^n K_1 \cup K_{p^n-1}] \). That is, the proper power graph of the dihedral group of order \( 2p^n \) (where \( n \geq 1 \)) can be expressed as the disjoint union of complete graphs.

Proof. In order to complete the proof, we consider both the cases where \( p \) is odd and \( p \) is even.

Assume that \( p \) is odd. For any odd prime \( p \), \( D_{p^n} \) contains exactly \( p^n \) elements of order 2 and so there are \( \frac{p^n}{\phi(2)} = p^n \) subgroups of order 2. Moreover, \( D_{p^n} \) has a subgroup isomorphic to \( \mathbb{Z}_{p^n} \). Thus, there are \( p^n \) copies of \( \mathbb{Z}_2 \) and a single copy of \( \mathbb{Z}_{p^n} \). By theorem 3.1, \( \mathcal{P}^*(\mathbb{Z}_{p^n}) \cong K_{p^n-1} \) and \( \mathcal{P}^*(\mathbb{Z}_2) \cong K_1 \). Thus, \( \mathcal{P}^*(D_{p^n}) \) contains \( p^n \) copies of \( K_1 \) and a single copy of \( K_{p^n-1} \). Hence, the result holds when \( p \) is odd.

Assume that \( p \) is even. Then, \( D_{2^n} \) contains a subgroup of order \( 2^n \) and \( 2^n + 1 \) elements of order 2. Out of these \( 2^n + 1 \) elements of order 2, one element lies in the cyclic group of order \( 2^n \). Thus, we need only count them as a single copy of \( \mathbb{Z}_{2^n} \) and \( 2^n \) copies of \( \mathbb{Z}_2 \). By theorem 3.1, \( \mathcal{P}^*(\mathbb{Z}_{2^n}) \cong K_{2^n-1} \) and \( \mathcal{P}^*(\mathbb{Z}_2) \cong K_1 \). Thus, there are \( 2^n \) copies of \( K_1 \) and a single copy of \( K_{2^n-1} \) (excluding the identity). Hence the result holds when \( p \) is even. Hence we get the desired result.

\[ \square \]

Theorem 3.6. For any distinct primes \( p \) and \( q \), \( \mathcal{P}^*(\mathbb{D}_{pq}) \) is not complete. Moreover,
\[ \mathcal{P}^*(\mathbb{D}_{pq}) \cong \{ [K_{p-1} \cup K_{q-1}] \lor K_{\phi(pq)} \} \cup pqK_1. \]

Proof. In order to complete the proof, we consider both the cases where \( p \) and \( q \) are odd and one of them is even.
Assume that \( p \) and \( q \) are odd. Then, \( D_{pq} \) contains a subgroup isomorphic to \( \mathbb{Z}_{pq} \) and \( pq \) elements of order 2. Thus there are \( pq \) copies of \( \mathbb{Z}_2 \) and a single copy of \( \mathbb{Z}_{pq} \). By theorem 3.1 and theorem 3.2, \( \mathcal{P}^*(\mathbb{Z}_2) \cong K_1 \) and \( \mathcal{P}^*(\mathbb{Z}_{pq}) \cong [K_{p-1} \cup K_{q-1}] \vee K_{\phi(pq)} \). Thus, \( \mathcal{P}^*(\mathbb{D}_{pq}) \) contains \( pq \) copies of \( K_1 \) and a single copy of \( [K_{p-1} \cup K_{q-1}] \vee K_{\phi(pq)} \). Hence the result holds when both are odd.

Assume that \( q \) is even. Then, \( \mathcal{P}^*(\mathbb{D}_{2p}) \) contains a subgroup of order \( \mathbb{Z}_{2p} \) and \( 2p + 1 \) elements of order 2. Out of these \( 2p + 1 \) elements of order 2, one element lies in the cyclic group of order \( 2p \). Thus, we need only count them as a single copy of \( \mathbb{Z}_{2p} \) and \( pq \) copies of \( \mathbb{Z}_2 \). By theorem 3.1 and theorem 3.2, \( \mathcal{P}^*(\mathbb{Z}_2) \cong K_1 \) and \( \mathcal{P}^*(\mathbb{Z}_{2p}) \cong [K_1 \cup K_{p-1}] \vee K_{\phi(2p)} \). Thus, there are \( 2p \) copies of \( K_1 \) and a single copy of \( [K_1 \cup K_{p-1}] \vee K_{\phi(2p)} \). Hence the result holds when one of them is even.

Hence we get the desired result. \( \square \)

**Theorem 3.7.** Let \( G \) be a finite cyclic group. Then \( \mathcal{P}^*(G) \) is complete if and only if \( G \cong \mathbb{Z}_{pn} \).

**Proof.** Assume that \( \mathcal{P}^*(G) \) is complete. Then for any two distinct vertices \( x, y \in G/\{e\} \), \( x^m = y \) or \( y = x^n \). That is, \( x \in \langle y \rangle \) or \( y \in \langle x \rangle \) since \( G \) is finite and cyclic, \( \gcd \) (greatest common divisor) of any two elements in \( G \) is well defined.

**Case 1** If \( \gcd(x, y) = 1 \).

Since \( x \in \langle y \rangle \) or \( y \in \langle x \rangle \) and \( \gcd(x, y) = 1 \), \( x \) is a generator or \( y \) is a generator. Thus, \( \langle y \rangle \) is a subgroup of \( \langle x \rangle \) or \( \langle x \rangle \) is a subgroup of \( \langle y \rangle \) \( \implies \) \( o(y) \mid o(x) \) or \( o(x) \mid o(y) \).

**Case 2** If \( \gcd(x, y) = d(\neq 1) \).

Since \( x \in \langle y \rangle \) or \( y \in \langle x \rangle \) and \( \gcd(x, y) = d(\neq 1) \), \( \langle y \rangle \) is a subgroup of \( \langle x \rangle \) or \( \langle x \rangle \) is a subgroup of \( \langle y \rangle \) \( \implies \) \( o(y) \mid o(x) \) or \( o(x) \mid o(y) \).

In either case, \( \langle y \rangle \) is a subgroup of \( \langle x \rangle \) or \( \langle x \rangle \) is a subgroup of \( \langle y \rangle \) \( \implies \) \( o(y) \mid o(x) \) or \( o(x) \mid o(y) \). Thus \( o(\langle x \rangle) \mid o(\langle y \rangle) \) or \( o(\langle y \rangle) \mid o(\langle x \rangle) \). Thus, \( o(\langle x \rangle) = p^{m_1} \) and \( o(\langle y \rangle) = p^{m_2} \) where \( m_1, m_2 \in \mathbb{Z} \) such that \( m_1 < m_2 \) or \( m_2 < m_1 \). Since order of an element divides the order of a group, \( o(G) = p^n.k \) where \( p \nmid k \).

Since \( p \nmid k \), \( k = q^m \). Then by Cauchy’s theorem, \( G \) has an element of order \( q \) say \( z \). then \( x \) and \( z \) are not adjacent (since \( z \) is not a power of \( p \)). Thus, \( G \) is not complete, a contradiction. Hence, \( G \) cannot have an element of order \( q \).

\( \therefore \) \( k = 1 \implies o(G) = p^n \). Since \( G \) is cyclic and \( o(G) = p^n \), then \( G \equiv \mathbb{Z}_{pn} \). Converse is true by theorem 3.1. Hence the result. \( \square \)

**Theorem 3.8.** \( \mathcal{P}^*(S_4) \cong 3K_3 \cup 4K_2 \cup 6K_1 \).
Proof. By theorem 3.8 of [5], we have $\mathcal{P}(S_4) \cong [3K_3 \cup 4K_2 \cup 6K_1] \vee K_1$. Applying Result 2.5, we have the desired result.

Theorem 3.9. $\mathcal{P}^*(S_5) \cong 10K_5 \cup 6K_4 \cup 15K_3$.

Proof. $S_5$ contains 24 elements of order 5, 30 elements of order 4, 20 elements of order 6, 20 elements of order 3 and 25 elements of order 2. Thus $S_5$ contains \( \frac{24}{\varphi(5)} = 6 \) subgroups of order 5, isomorphic to $Z_5$. Since $S_5$ contains an element of order 4, it has a cyclic subgroup of order 4. To be precise, there are \( \frac{30}{\varphi(4)} = 15 \) subgroups of order 4 isomorphic to $Z_4$. Each of these subgroups contains exactly one element of order 2, specifically elements of the form $(ab)(cd)(e)$ i.e, the product of two transpositions. Now, there are only remaining 10 elements of order 2.

Since $S_5$ contains an element of order 6, it has a cyclic subgroup of order 6. To be precise, there are \( \frac{20}{\varphi(6)} = 10 \) subgroups of order 6 (isomorphic to $Z_6$). Each copy of $Z_6$ contains exactly one element of order 2 and exactly two elements of order 3. Thus, the remaining 10 elements of order 2 and 20 elements of order 3 constitute 10 copies of $Z_6$.

Thus, there are six copies of $Z_5$, fifteen copies of $Z_4$, ten copies of $Z_6$. Thus,

\[
\mathcal{P}(S_5) \cong [10 \mathcal{P}(Z_6) \cup 6 \mathcal{P}(Z_5) \cup 15 \mathcal{P}(Z_4)] \vee K_1 \\
\cong [10K_5 \cup 6K_4 \cup 15K_3] \vee K_1.
\]

Applying Result 2.5, we get the desired result.

Theorem 3.10. $\mathcal{P}^*(A_4) \cong 4K_2 \cup 3K_1$.

Proof. By theorem 3.7 of [5], we have $\mathcal{P}(A_4) \cong [4K_2 \cup 3K_1] \vee K_1$. Applying Result 2.5, we get the desired result.

Theorem 3.11. $\mathcal{P}^*(A_5) \cong 6K_4 \cup 10K_2 \cup 15K_1$.

Proof. Elements in $A_5$ have orders 1, 2, 3, 5. There are 24 elements of order 5, 20 elements of order 3, 15 elements of order 2 and 1 element of order 1. Then by Result 2.2, there are \( \frac{24}{\varphi(5)} = 6 \) elements of order 5, \( \frac{20}{\varphi(3)} = 10 \) elements of order 3, \( \frac{15}{\varphi(2)} = 15 \) elements of order 2. Thus, there are 6 copies of $Z_5$, 10 copies of $Z_3$, 15 copies of $Z_2$. All these subgroups are disjoint except for the identity. Thus,

\[
\mathcal{P}(A_5) \cong [6 \mathcal{P}(Z_5) \cup 10 \mathcal{P}(Z_3) \cup 15 \mathcal{P}(Z_2)] \vee K_1.
\]

By Result 2.5, $\mathcal{P}^*(A_5) \vee K_1 \cong \mathcal{P}(A_5)$, the result follows.
4. Some Parameters Related to Proper Power Graphs

**Theorem 4.1.** For any prime $p$ and $n \geq 1$, $\Gamma(\mathcal{P}^*(\mathbb{Z}_{p^n})) \cong S_{p^n-1}$.

*Proof.* Let $G$ be any graph. Then, we know that if $G = K_p$ or $G = \overline{K}_p$, then $\Gamma(G)$ is $S_p$. Applying this to theorem 3.1, proof follows immediately. $\square$

**Theorem 4.2.** For any prime $p$ and $n \geq 1$, $\alpha(\mathcal{P}^*(\mathbb{Z}_{p^n})) = 1$, and $\beta(\mathcal{P}^*(\mathbb{Z}_{p^n})) = p^n - 2$.

*Proof.* The former result follows from the fact that $\alpha(K_n) = 1$ and using theorem 3.1, $\alpha(\mathcal{P}^*(\mathbb{Z}_{p^n})) = \alpha(K_{p^n-1}) = 1$. The latter part of the result follows immediately by result 2.6 $\square$

**Theorem 4.3.** For any prime $p$ and $n \geq 1$, $\alpha'(\mathcal{P}^*(\mathbb{Z}_{p^n})) = 1$, and $\beta'(\mathcal{P}^*(\mathbb{Z}_{p^n})) = p^n - 2$.

*Proof.* The former result follows from the fact that $\alpha'(K_n) = 1$ and using theorem 3.1, $\alpha'(\mathcal{P}^*(\mathbb{Z}_{p^n})) = \alpha'(K_{p^n-1}) = 1$. The latter part of the result follows immediately by result 2.7 $\square$

**Theorem 4.4.** For any two distinct primes $p$ and $q$, $\alpha(\mathcal{P}^*(\mathbb{Z}_{pq})) = 2$, and $\beta(\mathcal{P}^*(\mathbb{Z}_{pq})) = pq - 3$.

*Proof.* The former result follows from the fact that the set $\{p, q\}$ is a maximum independent set and the latter result follows immediately by result 2.6 $\square$

**Theorem 4.5.** For any prime $p$, $\alpha(\mathcal{P}^*(D_{p^n})) = p^n + 1$ where $D_{p^n}$ is the dihedral group of order $2p^n$ and $\beta(\mathcal{P}^*(D_{p^n})) = p^n - 2$.

*Proof.* The result follows from the fact that the set $S$ of all reflections together with any rotation constitute a maximum independent set. Thus $\alpha(\mathcal{P}^*(D_{p^n})) = |S| = p^n + 1$. The latter result follows immediately by result 2.6 $\square$

**Theorem 4.6.** For any two distinct primes $p$ and $q$, $\alpha(\mathcal{P}^*(D_{pq})) = pq + 2$, and $\beta(\mathcal{P}^*(D_{pq})) = pq - 3$.

*Proof.* The result follows from the fact that the set $S$ of all reflections together with two rotations constitute a maximum independent set. Thus $\alpha(\mathcal{P}^*(D_{pq})) = |S| = pq + 2$. The latter result follows immediately by result 2.6 $\square$
SOME CHARACTERIZATIONS OF PROPER POWER GRAPHS OF CERTAIN FINITE GROUPS

REFERENCES


DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE, THIRUVANANTHAPURAM
UNIVERSITY OF KERALA
Email address: umahgreen53@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE, THIRUVANANTHAPURAM
UNIVERSITY OF KERALA
Email address: manilalvarkala@gmail.com