SIGNED DOMINATION NUMBER OF n-STAR GRAPH

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ABSTRACT. The n-star graph $S_n$ is a simple graph whose vertex set is the set of all $n!$ permutations of \{1, 2, \ldots, n\} and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i, i \neq 1$. In this paper we find the signed domination number $\gamma_s$ of $S_n$. We also determine the lower bound of the signed domination number $\gamma_s$, for the complement of $S_n$, the lower bound of the sum and product of the signed domination number of n-star graph $S_n$ and its complement.

1. INTRODUCTION

By a graph we mean a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Haynes et al. [4] and Harary [3]. The n-star graph $S_n$ is first introduced by Akers and Krishnamurthy [1]. The vertex set of $S_n$ is the set of all $n!$ permutations of \{1, 2, \ldots, n\} and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i, i \neq 1$. In this paper we find the signed domination number for odd and even $n$ of $S_n$. We also obtain lower bound of signed domination number for the complement $S_n$ and sum and product of signed domination number of $S_n$ and its complement. Let $G=(V,E)$ be a graph. For a real valued function $f: V \rightarrow R$, the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$ and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$. A signed dominating function is defined as a function $f: V \rightarrow \{-1, 1\}$ such that $\sum_{u \in N[v]} f(u) \geq 1$, for all $v \in V$.

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The signed domination number for a graph $G$ is $\gamma_s(G) = \min \{w(f) | f \text{ is a signed dominating function on } G\}$. The upper signed domination number for a graph $\Gamma_s(G) = \max \{w(f) | f \text{ is a signed dominating function on } G\}$, [2,4].

**Theorem 1.1.** [4] For every $k$-regular graph $G$ of order $n$, $\gamma_s(G) \geq n/(k + 1)$.

**Theorem 1.2.** [4] For every $k$-regular graph $G$ of order $n$, with $k$ odd, $\gamma_s(G) \geq 2n/(k + 1)$.

2. **Main Results**

**Theorem 2.1.** For $n$-star graph, with $n$ odd the signed domination number $\gamma_s(S_n) = (n - 1)!$.

**Proof.**

**Case 1:** For $n = 1$.

Since there is only one vertex say $v(1)$ define $f : V(S_1) \rightarrow \{-1, 1\}$, such that $f(v_1) = 1$. Then $f$ is the only signed dominating function. The signed domination number of $S_1$ is $1 = (1 - 1)!$. Therefore $\gamma_s(S_n) = (n - 1)!$, for $n = 1$.

**Case 2:** For $n > 1$ and odd. $S_n$ is a $(n - 1)$ regular graph. Since $n$ is odd, $(n - 1)$ is even. We have for every $k$-regular graph $G$ of order $n$, $\gamma_s(G) \geq n/(k + 1)$. Therefore $S_n$,

$\gamma_s(S_n) \geq n/(n - 1 + 1) = (n - 1)!/n = (n - 1)!$.

Let $A_i = \{\alpha \in V(S_n) | \alpha(1) = 1, \alpha(1) = 2, \ldots \alpha(1) = i, \text{ where } i = (n - 1)/2\}$.

Define a function $f : V(S_n) \rightarrow \{-1, 1\}$ such that

$$f(\alpha) = \begin{cases} 
-1 & \text{for } \alpha \in A_i \\
1 & \text{for } \alpha \notin A_i 
\end{cases}$$

Then $f$ is a signed dominating function for $S_n$.

Now for finding the weight of signed dominating function for $S_n$, there are $(n - 1)!$ elements in each $\alpha(1) = i, i = 1, 2, \ldots, n$. Hence there are $(n - 1)!^2$ vertices for which $f(\alpha) = -1$. Therefore there are $n! - [(n - 1)!^2(n - 1)!]$ vertices for which $f(\alpha) = 1$. Then weight of the signed dominating function $f$ is

$$w(f) = \left[\frac{(n + 1)}{2}(n - 1)!\right] - \left[\frac{(n - 1)}{2}(n - 1)!\right] = \left[\frac{(n + 1)}{2} - \frac{(n - 1)}{2}\right] (n - 1)! = (n - 1)!$$.
Hence we get a signed dominating function $f$ of $S_n$ with weight $(n - 1)!$. But $\gamma_s(S_n) \geq (n - 1)!$.

Therefore $\gamma_s(S_n) = (n - 1)!$. \hfill \Box

**Theorem 2.2.** For n-star graph, the signed domination number, $\gamma_s(S_n) = 2(n-1)!$ where $n$ is even.

**Proof.**

**Case 1:** $n = 2$.
There are two vertices in $S_2$. Define $f : V(S_1) \rightarrow \{-1, 1\}$, such that $f(v_1) = 1$ and $f(v_2) = 1$. Then $f$ is the only signed dominating function. The signed domination number of $S_2$ is $2 = 2(2-1)!$. Therefore $\gamma_s(S_n) = 2(2-1)!$, for $n = 2$.

**Case 2:** $n > 2$ and even.
$S_n$ is a $(n - 1)$ regular graph. Since $n$ is even, $(n - 1)$ is odd. We have for every $k$-regular graph $G$ of order $n$, with $k$ odd, $\gamma_s(G) \geq 2n/(k + 1)$. Therefore for n-star graph,

$$\gamma_s(S_n) \geq 2n!/(n - 1 + 1) = 2(n - 1)!n/n = 2(n - 1)!.$$  

Let $A_i = \{\alpha \in V(S_n)|\alpha(1) = 1, \alpha(1) = 2, \ldots, \alpha(1) = i, \text{where } i = \frac{n}{2} - 1\}$. Define a function $f : V(S_n) \rightarrow \{-1, 1\}$, such that

$$f(\alpha) = \begin{cases} -1 & \text{for } \alpha \in A_i \\ 1 & \text{for } \alpha \notin A_i \end{cases}.$$  

Then $f$ is signed dominating function of $S_n$.

Now for finding the weight of signed dominating function for $S_n$, there are $(n - 1)!$ elements in each $\alpha(1) = i, i = 1, 2, \ldots, n$.

Hence there are $\left(\frac{n}{2} - 1\right)(n - 1)!$ vertices for which $f(\alpha) = -1$. Therefore there are $n! - \left[\left(\frac{n}{2} - 1\right)(n - 1)!\right] = \left(\frac{n}{2} + 1\right)(n - 1)!$ vertices for which $f(\alpha) = 1$. Then weight of the signed dominating function $f$ is

$$w(f) = \left[\left(\frac{n}{2} + 1\right)(n - 1)!\right] - \left[\left(\frac{n}{2} - 1\right)(n - 1)!\right] = \left[\left(\frac{n}{2} + 1\right) - \left(\frac{n}{2} - 1\right)\right](n - 1)! = 2(n - 1)!.$$  

Hence we get a signed dominating function $f$ of $S_n$ with weight $2(n - 1)!$. But $\gamma_s(S_n) \geq 2(n - 1)!$.

Hence $\gamma_s(S_n) = 2(n - 1)!$. \hfill \Box

**Theorem 2.3.** The complement $\overline{S_n}$ of n-star graph is $n! - n$ regular.
Proof. Any two vertices of $\overline{S_n}$ is adjacent if it is not adjacent in $S_n$. Clearly $S_n$ is $(n-1)$-regular. Also since there are $n! - 1$ vertices other than a vertex $v_i$ in $\overline{S_n}$, each vertex $v_i$ in $\overline{S_n}$ is adjacent with $(n! - 1) - (n-1) = n! - n$ vertices. Hence $\overline{S_n}$ is $n! - n$-regular.

**Theorem 2.4.** The signed domination number $\gamma(\overline{S_n}) \geq 2n!/(n! - n + 1)$ if $n$ is odd and $\gamma(\overline{S_n}) \geq n!/(n! - n + 1)$ if $n$ is even.

Proof. $\overline{S_n}$ is $n!-n$ regular. Also if $n$ is odd, then $n! - n$ is odd. We have for every k-regular graph $G$ of order $n$, with $k$ odd $\gamma(\overline{S_n}) \geq 2n/((k+1))$.

Therefore for the complement of $n$-star graph $\gamma(\overline{S_n}) \geq 2n!/(n! - n + 1)$.

Also if $n$ is even, then $(n! - n)$ is even. We have for every k-regular graph $G$ of order $n$, $\gamma(\overline{S_n}) \geq n/(k+1)$.

Therefore for the complement of $n$-star graph $\gamma(\overline{S_n}) \geq n!/(n! - n + 1)$.

**Theorem 2.5.** The sum of the signed domination number of $S_n$ and its complement $\overline{S_n}$ is

$$\gamma(S_n) + \gamma(\overline{S_n}) \geq \frac{(n-1)!(n! + n + 1)}{(n! - n + 1)},$$

for odd $n$.

Proof. By theorem 2.1, $\gamma(S_n) = (n-1)!$. Also by theorem 2.4, $\gamma(\overline{S_n}) \geq 2n!/(n! - n + 1)$, if $n$ is odd. Hence it follows that

$$\gamma(S_n) + \gamma(\overline{S_n}) \geq (n-1)! + 2n!/(n! - n + 1)$$

$$= \frac{(n-1)!(n! - n + 1) + 2(n-1)!n}{(n! - n + 1)} = \frac{(n-1)!(n! - n + 1 + 2n)}{(n! - n + 1)} = \frac{(n-1)!(n! + n + 1)}{(n! - n + 1)}.$$  

Hence $\gamma(S_n) + \gamma(\overline{S_n}) \geq \frac{(n-1)!(n! + n + 1)}{(n! - n + 1)}$, for $n$ odd.

**Theorem 2.6.** The sum of the signed domination number of $n$-star graph and its complement $\overline{S_n}$ is

$$\gamma(S_n) + \gamma(\overline{S_n}) \geq \frac{(n-1)!(2n - n + 1)}{(n! - n + 1)};$$

for even $n$. 

Proof. By theorem 2.2, $\gamma(S_n) = 2(n - 1)!$. Also by theorem 2.4, $\gamma(\overline{S}_n) \geq n!/(n! - n + 1)$. Hence it follows that

$$\gamma(S_n) + \gamma(\overline{S}_n) \geq 2(n - 1)! + n!/(n! - n + 1)$$

$$= 2(n - 1)!((n! - n + 1) + (n - 1)!n) = (n - 1)!(2n! - 2n + 1 + n)/(n! - n + 1)$$

$$= (n - 1)!(2n! - n + 1)/(n! - n + 1).$$

Hence $\gamma(S_n) + \gamma(\overline{S}_n) \geq (n - 1)!(2n! - n + 1)/(n! - n + 1)$ for $n$ even. □

Theorem 2.7. The product of the signed domination number of $n$-star graph and its complement is

$$\gamma(S_n)\gamma(\overline{S}_n) \geq \frac{(n - 1)!2n!}{(n! - n + 1)},$$

for $n$ odd and

$$\gamma(S_n)\gamma(\overline{S}_n) \geq \frac{2(n - 1)!n!}{(n! - n + 1)},$$

$n$ even.

Proof. By theorem 2.1, $\gamma(S_n) = (n - 1)!$ and by theorem 2.4, $\gamma(\overline{S}_n) \geq 2n!/(n! - n + 1)$ for odd $n$. Hence it follows that $\gamma(S_n)\gamma(\overline{S}_n) \geq (n - 1)!2n!/(n! - n + 1)$, for odd $n$. Also by theorem 2.2, $\gamma(S_n) = 2(n - 1)!$ and by theorem 2.5, $\gamma(\overline{S}_n) \geq n!/(n! - n + 1)$ for even $n$. Hence it follows that for even $n$

$$\gamma(S_n)\gamma(\overline{S}_n) \geq \frac{2(n - 1)!n!}{(n! - n + 1)}. □$$

References


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