IDEAL THEORY IN NEAR-SEMIRINGS AND ITS APPLICATION TO AUTOMATA

C. JENILA and P. DHEENA

Abstract. In this paper we develop ideal theory in near-semirings. We use the ideal theory to find the necessary and sufficient conditions for a linear sequential machine to be minimal.

1. Introduction

It has been shown that a homomorphic group-automaton \( A = (Q, A, B, F, G) \), where \( Q \) is a state set, \( A \) is an input set and \( B \) is an output set are groups and \( F : Q \times A \rightarrow Q \) and \( G : Q \times A \rightarrow B \), the state-transition function and output function respectively, are homomorphisms, is minimal if and only if the \( N(A) \)-group \( Q \) is generated by 0 and does not contain non-zero ideals which are annihilated by \( g_0 \) where \( g_0 : Q \rightarrow B \) ( [3], Theorem 9.259). Pilz [3] considered linear sequential machines in which the state set forms a group.

Krishna and Chatterjee [2] considered a generalized linear sequential machine \( \mathcal{M} = (Q, A, B, F, G) \) where \( Q, A, B \) are semigroups and \( R \)-semimodules for some semiring \( R \) and \( F : Q \times A \rightarrow Q \) and \( G : Q \times A \rightarrow B \) are \( R \)-homomorphisms. They have obtained a necessary condition for the above generalized sequential machine to be minimal. So naturally one is interested to find a necessary and sufficient conditions for the above generalized linear sequential machine to be minimal. To achieve that, we develop ideal theory in a

1corresponding author

2010 Mathematics Subject Classification. 16Y30, 16Y60.

Key words and phrases. Near-semiring, ideal, linear sequential machine.
$S$-semigroup $\Gamma$, where $S$ is a near-semiring. Using this ideal theory we find the necessary and sufficient conditions for a generalized linear sequential machine to be minimal. For the terminology and notation used in this paper we refer to Pilz [3], Krishna and Chatterjee [2].

2. Near-semirings

A near-semiring is a nonempty set $S$ with two binary operations $\cdot$ and $\cdot'$ such that

\begin{enumerate}
  \item $(S, \cdot)$ is a semigroup,
  \item $(S, \cdot')$ is a semigroup,
  \item $(x \cdot y)z = xz \cdot yz$ for all $x, y, z \in S$, and
  \item $0s = 0$ for all $s \in S$.
\end{enumerate}

In the near-semiring $(S, \cdot, \cdot')$, if $(S, \cdot)$ has identity then $S$ is a near-semiring with identity. Now we give a natural example of the near-semiring. Let $(\Gamma, \cdot)$ be a semigroup with identity $0$. If $M(\Gamma)$ is the set of all mappings from $\Gamma$ into $\Gamma$ then $M(\Gamma)$ is a near-semiring under pointwise addition and composition. $M(\Gamma)$ is neither a ring, nor a near-ring, nor a semiring. A semigroup $(S, \cdot)$ is an inverse semigroup if for each $a \in S$, there exists a unique element $a' \in S$ such that $a \cdot a' \cdot a = a$ and $a' \cdot a \cdot a' = a'$. Then $a'$ is the additive inverse of $a$. A near-semiring $(S, \cdot, \cdot')$ is an additive inverse near-semiring if $(S, \cdot')$ is an inverse semigroup. If $A$ and $B$ are any two non-empty sets of $S$, we define $AB = \{ab | a \in A, b \in B\}$. For $x, y \in S$, $x = (x')', (x+y)' = y' + x'$ and $(xy)' = x'y$. We have $E^+(S) = \{a \in S : a + a = a\}$.

The properties of additive inverse semiring were obtained by Bandelt and Petrich [1] and the properties of regularity in an additive inverse semiring were obtained by Sen and Maity [4]. They have assumed the three conditions.

\begin{enumerate}
  \item $a(a + a') = (a + a')$
  \item $a(b + b') = (b + b')a$
  \item $a + a(b + b') = a$.
\end{enumerate}

An element of $M(\Gamma)$ is said to be an affine mapping if it is a sum of an endomorphism and a constant map on $\Gamma$. The set of affine mappings on $\Gamma$ is a subsemigroup of $M(\Gamma)$, denoted by $M_{aff}(\Gamma)$. Throughout this paper $S$ denotes a near-semiring unless otherwise specified.
3. IDEAL THEORY

Now we develop ideal theory in a $S$-semigroup $\Gamma$.

**Definition 3.1.** Let $S$ be a near-semiring. A semigroup $(\Gamma, +)$ is said to be an $S$-semigroup if there exists a mapping $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \rightarrow \Gamma$ such that for all $x, y \in S$, $\gamma \in \Gamma$,

1. $(x + y)\gamma = x\gamma + y\gamma$,
2. $(xy)\gamma = x(y\gamma)$, and
3. $0\gamma = 0_{\Gamma}$, where $0_{\Gamma}$ is the zero of $\Gamma$.

**Definition 3.2.** A subsemigroup $\Delta$ of $S\Gamma$ with $S\Delta \subseteq \Delta$ is said to be an $S$-subsemigroup of $\Gamma$.

**Definition 3.3.** Let $S\Gamma_1$, $S\Gamma_2$ be $S$-semigroups. A map $f : S\Gamma_1 \rightarrow S\Gamma_2$ is called an $S$-homomorphism if $f(\gamma + \gamma_1) = f(\gamma) + f(\gamma_1)$ and $f(s\gamma) = sf(\gamma)$ for all $\gamma, \gamma_1 \in S\Gamma_1$ and $s \in S$.

Note that $f(0_{\Gamma_1}) = 0_{\Gamma_2}$.

**Definition 3.4.** If $f$ is an $S$-homomorphism of $\Gamma_1$ into $\Gamma_2$, then the kernel of $f$ is defined by $K = \{\gamma_1 \in \Gamma_1 | f(\gamma_1) = 0_{\Gamma_2}\}$.

Hereafter $(\Gamma, +)$ is assumed to be inverse semigroup with $E^+(\Gamma)$ in the center of $(\Gamma, +)$.

**Definition 3.5.** A non-empty subset $I$ of an $S$-semigroup $\Gamma$ is an ideal of $S\Gamma$ ($I \triangleleft_s \Gamma$) if

1. $E^+(\Gamma) \subseteq I$,
2. $i_1 + i_2 \in I$ for all $i_1, i_2 \in I$,
3. $\gamma + i + \gamma' \in I$ for all $\gamma \in \Gamma, i \in I$,
4. $s(i + \gamma) + (s\gamma)' \in I$ for all $\gamma \in \Gamma, i \in I$ and $s \in S$,
5. If $e + \gamma \in I$ implies $\gamma \in I$ for any $e \in E^+(\Gamma)$.

**Theorem 3.1.** If a non-empty subset $I$ of an $S$-semigroup $\Gamma$ satisfies the conditions (1), (2), (3), (4) and (5) given above then $I$ is the kernel of an $S$-homomorphism.

**Proof.** Define the relation $\rho$ on $\Gamma$ by $a \rho b$ for all $a, b \in \Gamma$ if and only if $i_1 + a = i_2 + b$ for some $i_1, i_2 \in I$. Clearly $\rho$ is reflexive and symmetric. Now we claim that $\rho$ is transitive. Assume that $a \rho b$ and $b \rho c$. Then $i_1 + a = i_2 + b$ and $i_3 + b = i_4 + c$ for
some \( i_1, i_2, i_3, i_4 \in I \). Now \( i_2 + i_3 + b = i_2 + i_4 + c \). Then \( i_2 + i_3 + b + b' + b = i_2 + i_4 + c \). Thus \( i_2 + b + b' + i_3 + b = i_2 + i_4 + c \). Hence \( i_1 + a + i_5 = i_2 + i_4 + c \) for some \( i_5 \in I \). Thus \( i_1 + a + a' + a + i_5 = i_2 + i_4 + c \). Then \( i_1 + a + i_5 + a' + a = i_2 + i_4 + c \). Thus \( i_1 + i_6 + a = i_2 + i_4 + c \) for some \( i_6 \in I \). Hence \( a pc \).

Let \( \Gamma / \rho = \{ [a] \mid a \in \Gamma \} \). Let us define \( + \) in \( \Gamma / \rho \) as \( [a] + [b] = [a + b] \) and the map \( S \times \Gamma / \rho \to \Gamma / \rho \) as \( s [a] = [sa] \) for all \( a, b \in \Gamma \) and \( s \in S \). Suppose that \( [a] = [a_1] \) and \( [b] = [b_1] \) for some \( a, a_1, b, b_1 \in \Gamma \). Then \( i_1 + a = i_2 + a_1 \) and \( i_3 + b = i_4 + b_1 \) for some \( i_1, i_2, i_3, i_4 \in I \). Now \( i_1 + a + i_3 + b = i_2 + a_1 + i_4 + b_1 \). Hence \( i_1 + a + i_3 + a' + a + b = i_2 + a_1 + i_4 + a' + a_1 + b_1 \). Then \( i_1 + i_5 + a + b = i_2 + i_6 + a_1 + b_1 \) for some \( i_5, i_6 \in I \). Thus, \( [a + b] = [a_1 + b_1] \).

Suppose that \( [a] = [a_1] \) for some \( a, a_1 \in \Gamma \). Then \( i_1 + a = i_2 + a_1 \) for some \( i_1, i_2 \in I \). Let \( s \in S \). Since \( s(i_1 + a) + (sa)' \in I \) and \( s(i_2 + a_1) + (sa_1)' \in I \), we have \( s(i_1 + a) + (sa)' + sa = i_3 + sa \) and \( s(i_2 + a_1) + (sa_1)' + sa_1 = i_4 + sa_1 \) for some \( i_3, i_4 \in I \). Let \( e = (sa)' + sa \) and \( e_1 = (sa_1)' + sa_1 \). Thus, \( s(i_1 + a) + e = i_3 + sa \) and \( s(i_2 + a_1) + e_1 = i_4 + sa_1 \). Since \( i_1 + a = i_2 + a_1 \), we have \( a_2 + e = i_3 + sa \) and \( a_2 + e_1 = i_4 + sa_1 \) for some \( a_2 = s(i_1 + a) \in \Gamma \). Therefore, \( a_2 + e + e_1 = i_5 + sa \) and \( a_2 + e + e_1 = i_6 + sa_1 \) for some \( i_5, i_6 \in I \). Thus, \( i_5 + sa = i_6 + sa_1 \). Hence \( [sa] = [sa_1] \). Thus, \( \Gamma / \rho \) is an \( S \)-semigroup.

Next we define \( \Psi : \Gamma \to \Gamma / \rho \) as \( \Psi(\gamma) = [\gamma] \), \( \gamma \in \Gamma \). Clearly \( \Psi \) is an \( S \)-homomorphism. Let \( K \) be the kernel. Take \( k \in K \). Then \( \Psi(k) = [0] \) implies \( [k] = [0] \) implies \( k \rho 0 \). Hence \( i_1 + k = i_2 + 0 \) for some \( i_1, i_2 \in I \). It follows that \( i_1 + k = i_2 \). Then \( i_1' + i_1 + k = i_1' + i_2 \). Let \( i_1' + i_2 = i_3 \). Hence \( i_1' + i_1 + k = i_3 \) implies \( i_1' + i_1 + k \in I \). Since \( i_1' + i_1 \in E^+(\Gamma) \), we have \( k \in I \). Therefore, \( K \subseteq I \). Clearly \( I \subseteq K \). Hence \( K = I \). Therefore, \( I \) is the kernel of an \( S \)-homomorphism. \( \square \)

4. Generalized linear sequential machine

**Definition 4.1.** A semiautomaton is a triple \( S = (Q, A, F) \), where \( Q \) is a state set, \( A \) is an input set and \( F : Q \times A \to Q \) is a state-transition function. If \( Q \) is an inverse semigroup (we always write it additively), we call \( S \) an inverse semigroup-semiautomaton and abbreviate this by ISA.

For \( q \in Q \) and \( a \in A \) we interpret \( F(q, a) \) as the new state obtained from the old state \( q \) by means of the input \( a \). We extend \( A \) to the free monoid \( A^* \) over \( A \) consisting of all finite sequences of elements of \( A \), including the empty sequence \( \Lambda \).
We define the function $f_a : Q \longrightarrow Q$ by
\[
\begin{align*}
    f_a(q) &= q, \\
    f_a(q) &= F(q, a) \text{ for all } a \in A, \\
    f_{xa}(q) &= F(f_x(q), a) \text{ for all } x \in A^*, a \in A.
\end{align*}
\]
Note that $f_{a_1a_2} = f_a f_{a_1}, a_1, a_2 \in A^*$.

Now we discuss two special cases.

**The homomorphism case:** Let $Q$ and $A$ be additive inverse semigroups with 0 and $F : Q \times A \longrightarrow Q$ be a homomorphism. Now $f_a(q) = F(q, a) = F((q, 0) + (0_Q, a)) = F(q, 0) + F(0_Q, a) = f_0(q) + f_a(0_Q)$. Hence $f_a = f_0 + \overline{f}_a$, where $f_0$ is a homomorphism (i.e., a distributive element in $M(Q)$), $\overline{f}_a$ is the map with constant value $f_a(0_Q)$. Then $S$ is called a homomorphic ISA.

**Proposition 4.1.** For $x = a_1a_2...a_n \in A^*$,
\[
f_x = f_0^n + (f_0^{n-1}\overline{f}_a + f_0^{n-2}\overline{f}_a + ... + f_0\overline{f}_{a_{n-1}} + \overline{f}_{a_n}),
\]
where $\overline{f}_a : Q \longrightarrow Q$ is the constant map with $\overline{f}_a(q) = f_a(0_Q)$ for all $q \in Q$.

**Proof.** We prove this result by induction on the length of the string $x$.

Let $a \in A$ and $q \in Q$. Now $f_a(q) = F(q, a) = F(q, 0) + F(0_Q, a) = f_0(q) + f_a(0_Q)$. Then $f_a = f_0 + \overline{f}_a$, so that the result is true for $n = 1$. Assume that the result is true for $n = k - 1$, i.e., $f_{a_1a_2...a_{k-1}} = f_0^{k-1} + (f_0^{k-2}\overline{f}_{a_1} + f_0^{k-3}\overline{f}_{a_2} + ... + f_0\overline{f}_{a_{k-2}} + \overline{f}_{a_{k-1}})$.

Now
\[
f_{a_1a_2...a_k} = f_a f_{a_1a_2...a_{k-1}} = (f_0 + \overline{f}_a)f_{a_1a_2...a_{k-1}} = f_0 f_{a_1a_2...a_{k-1}} + \overline{f}_a f_{a_1a_2...a_{k-1}}
\]
\[
= f_0(f_0^{k-1} + (f_0^{k-2}\overline{f}_{a_1} + f_0^{k-3}\overline{f}_{a_2} + ... + f_0\overline{f}_{a_{k-2}} + \overline{f}_{a_{k-1}})) + \overline{f}_a
\]
\[
= f_0^k + f_0^{k-1}\overline{f}_{a_1} + f_0^{k-2}\overline{f}_{a_2} + ... + f_0\overline{f}_{a_{k-1}} + \overline{f}_{a_k}.
\]
Hence the result by induction. \(\Box\)

**The linear case:** The linear case is a special case of the homomorphism case in which $Q$ and $A$ are $R$-semimodules for some semiring $R$ and $F$ is $R$-homomorphism.

Let $M = \{f_x | x \in A^*\}$. Clearly $M$ is a submonoid of $M_{aff}(Q)$. Note that $M_d = \{f_0^n | n \geq 1\}$ is the endomorphism part of $M$.

**Definition 4.2.** Let $S = (Q, A, F)$ be a ISA. The subnear-semiring $N(S)$ of $M_{aff}(Q)$ generated by $M$ is called the syntactic near-semiring of $S$. 
Theorem 4.1. Every non-zero element of \( N(S) \) can be written as \( \sum_{i=1}^{n} f_{x_i} \) for \( f_{x_i} \in M \).

Proof. Let \( f = \sum_{i=1}^{n} f_{x_i} \) and \( g = \sum_{j=1}^{m} f_{y_j} \) where \( f_{x_i}, f_{y_j} \in M \). Clearly \( N(S) \) is closed with respect to addition. Now

\[
fg = \left( \sum_{i=1}^{n} f_{x_i} \right) \left( \sum_{j=1}^{m} f_{y_j} \right) = \left( \sum_{i=1}^{n} (f_{x_i} + f_{y_i}) \right)
\]

\[
= \sum_{i=1}^{n} (f_{y_i} + f_{x_i}) + \sum_{j=1}^{m} f_{y_j}
\]

\[
= \sum_{i=1}^{n} (f_{x_i} + f_{y_i}) + \sum_{j=1}^{m} f_{y_j}
\]

\[
= \sum_{i=1}^{n} \left( f_{x_i} + f_{y_j} \right) = \sum_{i=1}^{n} f_{x_i} + \sum_{j=1}^{m} f_{y_j}.
\]

Since the above expression is a finite sum of elements of \( M, N(S) \) is closed with respect to multiplication. Hence the result. □

We extend \( A \) to the free near-semiring \( A^\# \) over \( A \). If \( a^\# = w(a_1, \ldots, a_n) \) is a word in \( A^\# \) we define \( f_w(a_1, \ldots, a_n) = f_{a_1} \cdots f_{a_n} \) and \( F^\#(q, a^\#) = f_{a^\#}(q) \). Thus, we get an extended simultaneous sequential ISA \( S^\# = (Q, A^\#, F^\#) \).

Definition 4.3. Let \( S = (Q, A, F) \) be an ISA and \( A^\# \) the free near-semiring on \( A \). \( q_1 \in Q \) is accessible from \( q_2 \in Q \) if there is some \( \alpha \in A^\# \) with \( f_{\alpha}(q_2) = q_1 \). \( S \) is accessible if each state \( q \) is accessible from each other state.

\( N(S) \) is not only a near-semiring, but it also operates on \( Q \).

Lemma 4.1. \( Q \) is an \( N(S) \)-inverse semigroup.

Proof. Define a map \( N(S) \times Q \rightarrow Q \) as for any \( n = \sum_{i=1}^{n} x_i, x_i \in M, q \in Q, (n, q) \mapsto nq \) which satisfies the following conditions:

\[
(1) \left( \sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j \right) q = \sum_{i=1}^{n} x_i(q) + \sum_{j=1}^{n} y_j(q), x_i, y_j \in M.
\]

\[
(2) \left( \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j \right) q = \sum_{i=1}^{n} x_i(\sum_{j=1}^{n} y_j(q)), x_i, y_j \in M.
\]

\[
(3) 0q = 0_q.
\]

□
Proposition 4.2. Let $S$ be an ISA. $S$ is accessible if and only if $Q$ is an $S = N(S)$-inverse semigroup with $S0_Q = Q$.

Proof. Assume that $S$ is accessible. Then $Q$ is an $N(S)$-inverse semigroup with $S0_Q = Q$. Conversely, suppose that $S0_Q = Q$. Let $q_1, q_2 \in Q$. Since $S0_Q = Q$, there exists $s \in S$ such that $s0_Q = q_1$. Now $s(0q_2) = q_1$. Then $(s0)q_2 = q_1$. Let $s0 = s_1 \in S$. Hence $s_1q_2 = q_1$. Therefore, $S$ is accessible. □

Definition 4.4. An automaton is a quintuple $A = (Q, A, B, F, G)$, where $(Q, A, F)$ is a semiautomaton, $B$ is an output set and $G : Q \times A \rightarrow B$ is an output function of $A$. If $Q$ is an inverse semigroup, $A$ is called an inverse semigroup-automaton and is denoted as IA.

$A$ is called a homomorphic IA if $Q, A, B$ are inverse semigroups and $F, G$ are homomorphisms. $A$ is called a linear IA or linear automaton or linear sequential machine if $Q, A, B$ are $R$-semimodules for some semiring $R$ and $F, G$ are $R$-homomorphisms.

Since for every automaton $A = (Q, A, B, F, G)$, $S = (Q, A, F)$ is a semiautomaton with the same attributes, we define $N(A)$ as $N(S)$.

5. IDEAL THEORY APPLIED TO MACHINES

Let $A^*$ and $B^*$ denote the free monoids over $A$ and $B$ respectively. For $q \in Q$, let $s_q : A^* \rightarrow B^*$ be defined by $s_q(\lambda) = \lambda$, $s_q(a) = G(q, a)$, $s_q(a_1a_2) = s_q(a_1)s_F(q, a_1)(a_2)$ and proceed inductively with $s_q(a_1a_2 \ldots a_n) = s_q(a_1a_2 \ldots a_{n-1})G(F(q, a_1 \ldots a_{n-1}), a_n)$.

Definition 5.1. $s_q : A^* \rightarrow B^*$ is called the sequential (input-output-) function of $A$ at $q$.

Define the relation $\sim$ on $Q$ by $q_1 \sim q_2$ if $s_{q_1} = s_{q_2}$ for all $q_1, q_2 \in Q$.

Proposition 5.1. Let $A$ be a linear IA. Then $\sim$ is a congruence relation in the $N(A)$-inverse semigroup $Q$.

Proof. Clearly $\sim$ is reflexive and symmetric. Assume that $q_1 \sim q_2$ and $q_2 \sim q_3$. Thus, $s_{q_1} = s_{q_2}$ and $s_{q_2} = s_{q_3}$, $q_1, q_2, q_3 \in Q$. Now $s_{q_1}(\lambda) = \lambda = s_{q_3}(\lambda)$, $s_{q_1}(a) = s_{q_3}(a)$ for all $a \in A$, $s_{q_1}(a_1a_2) = s_{q_1}(a_1)G(F(q_1, a_1), a_2) = s_{q_3}(a_1)G(F(q_3, a_1), a_2) = s_{q_3}(a_1a_2)$
for all \( a_1, a_2 \in A \), and so on.

Hence \( s_{q_1} = s_{q_2} \). Therefore, \( q_1 \sim q_3 \). Thus, \( \sim \) is transitive.

If \( q_1 \sim q_2 \) then \( s_{q_1} = s_{q_2} \). Let \( q \in Q \). Then \( s_{q_1 + q}(\wedge) = \wedge = s_{q_2 + q}(\wedge) \).

Let \( a \in A \). Now

\[
q_{q_1 + q}(a) = G(q_1 + q, a) = G(q_1, a) + G(q, a') + G(0, q, a)
\]

\[
= G(q_2, a) + G(q, a') + G(0, q, a) = G(q_2 + q, a) = s_{q_2 + q}(a).
\]

Let \( a_1, a_2 \in A \). Now

\[
s_{q_1 + q}(a_1 a_2) = s_{q_1 + q}(a_1)G(F(q_1 + q, a_1, a_2)
\]

\[
= s_{q_1 + q}(a_1)G((F(q_1, a_1), a_2) + (F(q, a'_1), a'_2) + (F(0, q, a_1), a_2))
\]

\[
= s_{q_2 + q}(a_1)(F(q_2, a_1, a_2) + (F(q, a'_1), a'_2) + (F(0, q, a_1), a_2))
\]

\[
= s_{q_2 + q}(a_1)G(F(q_2 + q, a_1), a_2 = s_{q_2 + q}(a_1 a_2),
\]

and so on. Hence \( s_{q_1 + q} = s_{q_2 + q} \). Thus, \( q_1 + q \sim q_2 + q \).

Let \( a \in A \) and \( n = f_{a_1 a_2 \ldots a_k} \in N(A) \). Suppose that \( q_1 \sim q_2 \). Now,

\[
s_{nq_1}(a) = G(nq_1, a) = G(f_{a_1 a_2 \ldots a_k}(q_1), a)
\]

\[
= G(F(q_1, a_1 a_2 \ldots a_k), a) = G(F(q_2, a_1 a_2 \ldots a_k), a)
\]

\[
= G(f_{a_1 a_2 \ldots a_k}(q_2), a) = s_{nq_2}(a).
\]

Assume that \( s_{nq_1}(a_1 a_2 \ldots a_{n-1}) = s_{nq_2}(a_1 a_2 \ldots a_{n-1}) \). Now,

\[
s_{nq_1}(a_1 a_2 \ldots a_n) = s_{nq_1}(a_1 a_2 \ldots a_{n-1})G(F(nq_1, a_1 a_2 \ldots a_{n-1}, a_n)
\]

\[
= s_{nq_2}(a_1 a_2 \ldots a_{n-1})G(F(f_{a_1 a_2 \ldots a_k}(q_1), a_1 a_2 \ldots a_{n-1}, a_n)
\]

\[
= s_{nq_2}(a_1 a_2 \ldots a_{n-1})G(F(F(q_1, a_1 a_2 \ldots a_k), a_1 a_2 \ldots a_{n-1}, a_n)
\]

\[
= s_{nq_2}(a_1 a_2 \ldots a_{n-1})G(F(F(q_2, a_1 a_2 \ldots a_k), a_1 a_2 \ldots a_{n-1}, a_n)
\]

\[
= s_{nq_2}(a_1 a_2 \ldots a_{n-1})G(F(F(nq_2, a_1 a_2 \ldots a_{n-1}, a_n)
\]

\[
= s_{nq_2}(a_1 a_2 \ldots a_n).
\]

By induction, \( s_{nq_1} = s_{nq_2} \). Hence \( nq_1 \sim nq_2 \).

Let \( Q_0 = \{ q \in Q | q \sim 0 \} \). Hereafter we assume that \( e + q = q + e \) for all \( e \in E^+(Q) \), \( q \in Q \) and \( E^+(Q) \subseteq Q_0 \). If \( Q \) is a group, the above conditions are trivially satisfied.

**Theorem 5.1.** If \( A \) is a linear IA then:

1. \( Q_0 = \{ q \in Q | q \sim 0 \} \trianglelefteq N(A) \) \( Q \);
2. \( G(q, 0) = 0_B \) for all \( q \in Q_0 \).

**Proof.**

1. Let \( q_1, q_2 \in Q_0 \). Then \( q_1 \sim 0 \) and \( q_2 \sim 0 \). Since \( q_2 \sim 0 \), we have \( q'_1 + q_2 \sim q'_2 \). Thus, \( q'_2 \sim q'_2 + q_2 \in E^+(Q) \subseteq Q_0 \) implies \( q'_2 \sim 0 \). Hence \( q_1 + q'_2 \sim 0 \). Let \( q \in Q \) and
Let \( \mathcal{A} = (Q, A, B, F, G) \) be a linear IA and \( g_0 : Q \to B, q \mapsto g_0(q) = G(q, 0) \). If \( (g_0 f^k_0)(q) = (g_0 f^k)(q_1) \) for all \( k \geq 0 \) then \( q \sim q_1 \).

**Proof.** We prove this result by induction on the length of the string \( a \in A^* \). If \( k = 0 \) then \( G(q, 0) = G(q_1, 0) \) for all \( q, q_1 \in Q \). Let \( a \in A \).

Now, \( s_q(a) = G(q, a) = G(q, 0) + G(0Q, a) = G(q_1, 0) + G(0Q, a) = G(q_1, a) = s_{q_1}(a) \). Assume the result is true for \( k-1 \), i.e. \( s_q(a_1a_2\ldots a_{k-1}) = s_{q_1}(a_1a_2\ldots a_{k-1}) \).

Then
\[
G(f_{a_1a_2\ldots a_{k-1}}(q), a_k) = G\left((f_0^{k-1} + (f_0^{k-2} f_1 + \cdots + f_{a_{k-1}}))(q), a_k\right)
\]

\[
= G(f_0^{k-1}(q), 0) + G\left((f_0^{k-2} f_1 + \cdots + f_{a_k})(q), 0\right) + G(0Q, a_k)
\]

\[
= G(f_0^{k-1}(q_1), 0) + G(f_0^{k-2} f_1 + \cdots + f_{a_{k-1}}(q_1), 0) + G(0Q, a_k)
\]

\[
= G(f_{a_1a_2\ldots a_{k-1}}(q_1), a_k).
\]

Now,
\[
s_q(a_1a_2\ldots a_k) = s_q(a_1a_2\ldots a_{k-1})G(F(q, a_1a_2\ldots a_k), a_k)
\]

\[
= s_{q_1}(a_1a_2\ldots a_{k-1})G(f_{a_1a_2\ldots a_{k-1}}(q), a_k)
\]

\[
= s_{q_1}(a_1a_2\ldots a_{k-1})G(f_{a_1a_2\ldots a_{k-1}}(q_1), a_k)
\]

\[
= s_{q_1}(a_1a_2\ldots a_k).
\]

Hence \( q \sim q_1 \). \( \square \)

**Definition 5.2.** An IA \( \mathcal{A} = (Q, A, B, F, G) \) is reduced if \( \sim \) is the equality. If \( \mathcal{A} \) is accessible (i.e. if \( (Q, A, F) \) is accessible) and reduced then \( \mathcal{A} \) is called minimal.

**Theorem 5.3.** Let \( \mathcal{A} \) be a linear IA. Then \( \mathcal{A} \) is reduced if and only if \( N(\mathcal{A})Q \) has no non-zero ideals \( P \) with \( g_0P = \{0_B\} \).

**Proof.** Assume that \( N(\mathcal{A})Q \) has no such ideals. By Theorem 5.1, \( Q_0 \) is an ideal of \( N(\mathcal{A})Q \) with \( g_0Q_0 = \{0_B\} \). Then \( Q_0 = \{0\} \). Hence \( \mathcal{A} \) is reduced.

Conversely suppose that \( \mathcal{A} \) is reduced and that \( P \subseteq N(\mathcal{A}) \) has \( g_0P = \{0_B\} \). Then \( G(p, 0) = g_0(p) = 0_B \) for all \( p \in P \). Since \( f_0^k(p + 0) = (f_0^k(0))^\prime \in P \) for all
$p \in P$, we have $f_k^0(p) \in P$. Then $(g_0 f_k^0)(p) = 0_B$ for all $p \in P, k \geq 0$. Therefore, 
$(g_0 f_k^0)(p) = 0_B = (g_0 f_0^0)(0_Q)$ for all $k \geq 0$. Thus, $p \sim 0_Q$ by Theorem 5.2. Hence 
$p = 0_Q$. Then $P = \{0_Q\}$. □

From Proposition 4.2 and Theorem 5.3 we get

**Theorem 5.4.** Let $A$ be a linear IA. Then $A$ is minimal if and only if $N(A)Q$ is zero 
generated and does not contain non-zero ideals which are annihilated by $g_0$.

Thus, in an Automata, if $Q$ is not necessarily group but inverse semigroup, we 
have extended the result obtained for group Automata to check the minimality.

**References**


Department of Mathematics
Holy Cross College (Autonomous)
Nagercoil - 629 004, India
*Email address: jenilac201@gmail.com*

Department of Mathematics
Annamalai University
Annampalainagar - 608 002, India
*Email address: dheenap@yahoo.com*