THE HYPER-ZAGREB INDEX OF SOME COMPLEMENT GRAPHS

MOHAMMED SAAD ALSHARAFI, MAHIOUB MOHAMMED SHUBATAH, AND ABDU QAID ALAMERI

ABSTRACT. In this study, the Hyper-Zagreb index for some complement graphs operations has been computed, that have been applied to compute the Hyper-Zagreb index for complement molecular graph of a nanotorus and titania nanotubes.

1. INTRODUCTION

Mathematical chemistry is a branch of theoretical chemistry in which we use mathematical methods to analyze and predict the chemical structure. Chemical graph theory is a branch of mathematical chemistry where we use tools from graph theory to mathematically model the chemical phenomenon. This theory plays an important role in Function in the Chemical Sciences [9]. Throughout this paper, we consider a finite connected graph \( G \) that has no loops or multiple edges with vertex and edge sets \( V(G) \), and \( E(G) \), respectively. For a graph \( G \), the degree of a vertex \( u \) is the number of edges incident to \( u \), denoted by \( \delta_G(u) \). The complement of \( G \), denoted by \( \overline{G} \), is a simple graph on the same set of vertices \( V(G) \) in which two vertices \( u \) and \( v \) are adjacent, i.e., connected by an edge \( uv \), if and only if they are not adjacent in \( G \). Hence, \( uv \in E(\overline{G}) \), if and only if \( uv \notin E(G) \). Obviously \( E(G) \cup E(\overline{G}) = E(K_n) \), and \( m = |E(\overline{G})| = 

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\( \binom{n}{2} - m \), the degree of a vertex \( u \) in \( G \), is the number of edges incident to \( u \), denoted by \( \delta_G(u) = n - 1 - \delta_G(v) \) [11]. The first and second Zagreb indices have been introduced by Gutman and Trinajstic in 1972 [10]. They are respectively defined as:
\[
M_1(G) = \sum_{v \in V(G)} \delta_G^2(v) = \sum_{uv \in E(G)} [\delta_G(u) + \delta_G(v)], \quad M_2(G) = \sum_{uv \in E(G)} \delta_G(u) \delta_G(v),
\]

In 2013, G.H. Shirdel, H. Rezapour and A.M. Sayadi [5] introduced distance-based Zagreb indices named Hyper-Zagreb index which is defined as:
\[
HM(G) = \sum_{uv \in E(G)} (\delta_G(u) + \delta_G(v))^2.
\]

Furtula and Gutman in 2015 introduced forgotten index (F-index) [4] which defined as:
\[
F(G) = \sum_{uv \in E(G)} (\delta_G^2(u) + \delta_G^2(v)).
\]

In 2020, computed exact formulas for the Y-index of some graph operations by A. Alameri et al [1]. They defined a new distance-based of forgotten indices named Yemen-index (Y-index) defined as:
\[
Y(G) = \sum_{u \in V(G)} \delta_G^4(u) = \sum_{uv \in E(G)} [\delta_G^3(u) + \delta_G^3(v)].
\]

2. Preliminaries

In this section we give some basic and preliminary concepts which we shall use later.

Lemma 2.1. [2] Let \( G_1 \) and \( G_2 \) be two connected graphs with \( |V(G_1)| = n_1, |V(G_2)| = n_2, |E(G_1)| = m_1, \) and \( |E(G_2)| = m_2. \) Then
\[
(1) \quad |V(G_1 \times G_2)| = |V(G_1 \lor G_2)| = |V(G_1 \circ G_2)| = |V(G_1 \otimes G_2)| = |V(G_1 * G_2)| = |V(G_1 \oplus G_2)| = |V(G_1 + G_2)| = n_1n_2,
\]
\[
(2) \quad |E(G_1 \times G_2)| = m_1n_2 + n_1m_2, \quad |E(G_1 \lor G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2, \quad |E(G_1 \circ G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2, \quad |E(G_1 \otimes G_2)| = m_1n_2^2 + m_2n_1, \quad |E(G_1 \oplus G_2)| = m_1n_2^2 + m_2n_1^2 - 2m_1m_2, \quad |E(G_1 + G_2)| = 2m_1m_2, \quad |E(G_1 \times G_2)| = m_1n_2 + m_2n_1^2 - 4m_1m_2.
\]
Corollary 2.1. [10] The first Zagreb index of some well-known graphs: For path graph $P_n$ and cycle graph $C_n$, with $n : n \geq 3$ vertices:

$$M_1(C_n) = 4n, \quad M_1(P_n) = 4n - 6.$$  

Corollary 2.2. [3, 5] The Hyper-Zagreb index of some well-known graphs: For path $P_n$ and cycle graphs $C_n$, with $n, m \geq 3$ vertices:

$$HM(C_n) = 16n, \quad HM(P_n) = 16n - 30, \quad M(P_n \times C_m) = 128nm - 150m.$$  

Theorem 2.1. [6] Let $G$ be a simple graph on $n$ vertices and $m$ edges. Then:

$$M_1(G) = n(n-1)^2 - 4m(n-1) + M_1(G),$$

$$HM(G) = 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n - 6)M_1(G) - HM(G).$$  

3. MAIN RESULTS

In this section, we study the Hyper-Zagreb index of various complement graph binary operations such as Cartesian product $G_1 \times G_2$, composition $G_1 \circ G_2$, disjunction $G_1 \lor G_2$, symmetric difference $G_1 \oplus G_2$, join $G_1 + G_2$, tensor product $G_1 \otimes G_2$, and strong product $G_1 \ast G_2$, of graphs. We use the notation $V(G_i)$ for the vertex set, $E(G_i)$ for the edge set, $n_i$ for the number of vertices and $m_i$, $\overline{m}_i$ for the number of edges of the graph $G_i$, $\overline{G}_i$ respectively. All graphs here offer are simple graphs.

Proposition 3.1. The Hyper-Zagreb index of the complement of $(G_1 \otimes G_2)$ is given by:

$$HM(\overline{G}_1 \otimes \overline{G}_2)$$

$$= 2n_1n_2(n_1n_2 - 1)^3 - 24m_1m_2(n_1n_2 - 1)^2 + 16m_1^2m_2^2 + (5n_1n_2 - 6)M_1(G_1)M_1(G_2) - F(G_1)F(G_2) + 2M_2(G_1)M_2(G_2).$$  

Proof. By using Theorem 2.1 we have:

$$HM(\overline{G}_1 \otimes \overline{G}_2)$$

$$= 2|V(G_1 \otimes G_2)||V(G_1 \otimes G_2)| - 1)^3 - 12|E(G_1 \otimes G_2)||V(G_1 \otimes G_2)| - 1)^2$$

$$+ 4|E(G_1 \otimes G_2)|^2 + (5|V(G_1 \otimes G_2)| - 6)M_1(G_1 \otimes G_2) - HM(G_1 \otimes G_2).$$
By Lemma 2.1 \(|E(G_1 \otimes G_2)| = 2m_1m_2, \quad |V(G_1 \otimes G_2)| = n_1n_2,\) and by [7] and [8], respectively, we have

\[M_1(G_1 \otimes G_2) = M_1(G_1)M_1(G_2), \quad HM(G_1 \otimes G_2) = F(G_1)F(G_2) + 2M_2(G_1)M_2(G_2),\]

which is complete the proof. \qed

**Proposition 3.2.** The Hyper-Zagreb index of the complement of \((G_1 + G_2)\) is given by:

\[
HM(G_1 + G_2) = 2n_1n_2(n_1n_2 - 1)^3 - 12(m_1 + m_2 + n_1n_2)(n_1n_2 - 1)^2 + 4(m_1 + m_2 + n_1n_2)^2
\]
\[+ (5n_1n_2 - 6)[M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1]
\[- [HM(G_1) + HM(G_2) + 5(n_1M_1(G_2) + n_2M_1(G_1))
\[+ 8[n_1^2m_2 + n_2^2m_1 + m_1m_2] + n_1n_2[(n_2 + n_1)^2 + 4(m_1 + m_2)]].
\]

**Proof.** By using Theorem 2.1 we have

\[
HM(G_1 + G_2) = 2|V(G_1 + G_2)||(|V(G_1 + G_2)| - 1)^3 - 12|E(G_1 + G_2)||(|V(G_1 + G_2)| - 1)^2
\[+ 4|E(G_1 + G_2)|^2 + (5|V(G_1 + G_2)| - 6)M_1(G_1 + G_2) - HM(G_1 + G_2),
\]

By Lemma 2.1 \(|E(G_1 + G_2)| = m_1 + m_2 + n_1n_2, \quad |V(G_1 + G_2)| = n_1n_2,\) and by [7] and [5], respectively, we have

\[M_1(G_1 + G_2) = M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1,
\]

\[
HM(G_1 + G_2) =
\]
\[HM(G_1) + HM(G_2) + 5(n_1M_1(G_2) + n_2M_1(G_1))
\[+ 8[n_1^2m_2 + n_2^2m_1 + m_1m_2] + n_1n_2[(n_2 + n_1)^2 + 4(m_1 + m_2)],
\]

which is complete the proof. \qed

**Proposition 3.3.** Let \(G_1, G_2\) be two simple connected graphs with \(n_1, n_2\) vertices and \(m_1, m_2\) edges, respectively. Then,

\[M_1(G_1 \ast G_2) = (n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2).
\]
Proposition 3.4. The Hyper-Zagreb index of the complement of \((G_1 \ast G_2)\) is given by:

\[
HM(G_1 \ast G_2) = 2n_1n_2(n_1n_2 - 1)^3 - 12(m_1n_2 + n_1m_2 + 2m_1m_2)(n_1n_2 - 1)^2 + 4(m_1n_2 + n_1m_2 + 2m_1m_2)^2 + (5n_1n_2 - 6)[(n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2) - HM(G_1) + n_1HM(G_2) + 5n_2M_1(G_1) + 5n_1M_1(G_2) + 4n_2m_1[2n_2 + 1] + 8m_2[n_1 + m_1] + n_1n_2(n_3^2 + 2n_2 + 4m_2)]
\]

Proof. By using Theorem 2.1 we have

\[
HM(G_1 \ast G_2) = 2|V(G_1 \ast G_2)|(|V(G_1 \ast G_2)| - 1)^3 - 12|E(G_1 \ast G_2)|(|V(G_1 \ast G_2)| - 1)^2 + 4|E(G_1 \ast G_2)|^2 + (5|V(G_1 \ast G_2)| - 6)M_1(G_1)M_1(G_2) - HM(G_1 \ast G_2).
\]

By Lemma 2.1 \(|E(G_1 \ast G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2\), \(|V(G_1 \ast G_2)| = n_1n_2\), and by Proposition 3.3 and [5], respectively, we have

\[
M_1(G_1G_2) = (n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2),
\]

\[
HM(G_1 \ast G_2) = HM(G_1) + n_1HM(G_2) + 5n_2M_1(G_1) + 5n_1M_1(G_2) + 4n_2m_1[2n_2 + 1] + 8m_2[n_1 + m_1] + n_1n_2(n_3^2 + 2n_2 + 4m_2),
\]

which is complete the proof. \(\square\)

Proposition 3.5. The Hyper-Zagreb index of the complement of \((G_1 \times G_2)\) is given by:

\[
HM(G_1 \times G_2) = 2n_1n_2(n_1n_2 - 1)^3 - 12(m_1n_2 + m_2n_1)(n_1n_2 - 1)^2 + 4(m_1n_2 + m_2n_1)^2 + (5n_1n_2 - 6)[n_2M_1(G_1) + n_1M_1(G_2) + 8m_2m_1] - n_2HM(G_1) + n_1HM(G_2) + 12m_1M_1(G_2) + 12m_2M_1(G_1)].
\]
Proof. By using Theorem 2.1 we have

\[
\text{HM}(G_1 \times G_2) \\
= 2|V(G_1 \times G_2)|(|V(G_1 \times G_2)| - 1)^3 - 12|E(G_1 \times G_2)||V(G_1 \times G_2)| - 1)^2 \\
+ 4|E(G_1 \times G_2)|^2 + (5|V(G_1 \times G_2)| - 6)M_1(G_1 \times G_2) - \text{HM}(G_1 \times G_2).
\]

By Lemma 2.1 \(|E(G_1 \times G_2)| = m_1 n_2 + n_1 m_2, \quad |V(G_1 \times G_2)| = n_1 n_2, and by [7] and [3] respectively, we have

\[
M_1(G_1 \times G_2) = n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2,
\]

\[
\text{HM}(G_1 \times G_2) = n_2 \text{HM}(G_1) + n_1 \text{HM}(G_2) + 12m_1 M_1(G_2) + 12m_2 M_1(G_1),
\]

which is complete the proof. \(\square\)

Proposition 3.6. The Hyper-Zagreb index of the complement of \((G_1 \circ G_2)\) is given by:

\[
\text{HM}(\overline{G_1 \circ G_2}) \\
= 2n_1 n_2(n_1 n_2 - 1)^3 - 12[m_1 n_2^2 + m_2 n_1](n_1 n_2 - 1)^2 + 4[m_1 n_2^2 + m_2 n_1]^2 \\
+ (5n_1 n_2 - 6)[n_2^3 M_1(G_1) + n_1 M_1(G_2) + 8n_2 m_2 m_1] - [n_2^4 \text{HM}(G_1) \\
+ n_1 \text{HM}(G_2) + 12n_2^2 m_2 M_1(G_1) + 10n_2 m_1 M_1(G_2) + 8m_2 m_1].
\]

Proof. By using Theorem 2.1 we have

\[
\text{HM}(\overline{G_1 \circ G_2}) \\
= 2|V(G_1 \circ G_2)|(|V(G_1 \circ G_2)| - 1)^3 - 12|E(G_1 \circ G_2)||V(G_1 \circ G_2)| - 1)^2 \\
+ 4|E(G_1 \circ G_2)|^2 + (5|V(G_1 \circ G_2)| - 6)M_1(G_1 \circ G_2) - \text{HM}(G_1 \circ G_2),
\]

By Lemma 2.1 \(|E(G_1 \circ G_2)| = m_1 n_2^2 + m_2 n_1, \quad |V(G_1 \circ G_2)| = n_1 n_2, and by [10] and [3] respectively, we have

\[
M_1(G_1 \circ G_2) = n_2^3 M_1(G_1) + n_1 M_1(G_2) + 8n_2 m_2 m_1,
\]

\[
\text{HM}(G_1 \circ G_2) = n_2^4 \text{HM}(G_1) + n_1 \text{HM}(G_2) + 12n_2^2 m_2 M_1(G_1) + 10n_2 m_1 M_1(G_2) + 8m_2 m_1,
\]

which is complete the proof. \(\square\)
**Proposition 3.7.** The Hyper-Zagreb index of the complement of \((G_1 \vee G_2)\) is given by:

\[
HM(G_1 \vee G_2) = 2n_1n_2(n_1n_2 - 1)^3 - 12[n_1n_2^2 + m_2n_1^2 - 2m_1m_2](n_1n_2 - 1)^2 \\
+ 4[n_1n_2^2 + m_2n_1^2 - 2m_1m_2]^2 + (5n_1n_2 - 6)[(n_1n_2^2 - 4m_2n_2)M_1(G_1) \\
+ M_1(G_2)M_1(G_1) + (n_2n_1^2 - 4m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2] \\
- [[n_1^2 - 2n_2^2m_2]HM(G_2) + [n_1^2 - 2n_2^2m_2]HM(G_1) + 5n_1M_1(G_1)F(G_2) \\
+ 5n_2M_1(G_2)F(G_1) + 10n_2^2m_2n_1M_1(G_1) + 10n_2n_1^2m_1M_1(G_2) \\
+ 8n_2^2m_2m_1 + 8n_1^2m_1m_2 - 8n_2m_1^2M_1(G_2) - 8n_1m_2^2M_1(G_1) \\
- 4n_1^2m_1F(G_2) - 4n_2^2m_2F(G_1) - 8n_1^2m_1M_2(G_2) - 8n_2^2m_2M_2(G_1) \\
+ 8n_1M_2(G_2) + 8n_2M_2(G_1) - 8n_2m_1M_1(G_1)M_1(G_2) + 4n_2M_2(G_1)M_1(G_2) \\
+ 4n_1M_2(G_2)M_1(G_1) - 2F(G_1)F(G_2) - 4M_2(G_2)M_2(G_2)].
\]

**Proof.** By using Theorem 2.1 we have

\[
HM(G_1 \vee G_2) = 2|V(G_1 \vee G_2)||\left|V(G_1 \vee G_2)\right| - 1)^3 - 12|E(G_1 \vee G_2)||\left|V(G_1 \vee G_2)\right| - 1)^2 \\
+ 4|E(G_1 \vee G_2)|^2 + (5|V(G_1 \vee G_2)| - 6)M_1(G_1 \vee G_2) - HM(G_1 \vee G_2),
\]

By Lemma 2.1 \(|E(G_1 \vee G_2)| = m_1n_2^2 + m_2n_1^2 - 2m_1m_2, \quad |V(G_1 \vee G_2)| = n_1n_2,\) and by [10] and [8] respectively, we have

\[
M_1(G_1 \vee G_2) = (n_1n_2^2 - 4m_2n_2)M_1(G_1) + M_1(G_2)M_1(G_1) \\
+ (n_2n_1^2 - 4m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2.
\]

\[
HM(G_1 \vee G_2) = n_1^2 - 2n_2^2m_2]HM(G_2) + [n_1^2 - 2n_2^2m_2]HM(G_1) + 5n_1M_1(G_1)F(G_2) \\
+ 5n_2M_1(G_2)F(G_1) + 10n_2^2m_2n_1M_1(G_1) + 10n_2n_1^2m_1M_1(G_2) \\
+ 8n_2^2m_2m_1 + 8n_1^2m_1m_2 - 8n_2m_1^2M_1(G_2) - 8n_1m_2^2M_1(G_1) - 4n_1^2m_1F(G_2) \\
- 4n_2^2m_2F(G_1) - 8n_1^2m_1M_2(G_2) - 8n_2^2m_2M_2(G_1) + 8n_1M_2(G_2) + 8n_2M_2(G_1) - 8n_2m_1M_1(G_1)M_1(G_2) + 4n_2M_2(G_1)M_1(G_2) \\
+ 4n_1M_2(G_2)M_1(G_1) - 2F(G_1)F(G_2) - 4M_2(G_2)M_2(G_2)].
\]
by:

The hyper-Zagreb index of complement nanotube Figure 1 is given by

\[ HM(G_1 \oplus G_2) = 2n_1n_2(n_1n_2 - 1)^3 - 12[m_1n_2^3 + m_2n_1^3 - 4m_1m_2(n_1n_2 - 1)^2
+4[m_1n_2^2 + m_2n_1^2 - 4m_1m_2]^2 + (5n_1n_2 - 6)(n_1n_2^3 - 8m_2n_2)M_1(G_1)
+4M_1(G_1)M_1(G_2) + (n_2m_1^2 - 8m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2
-[(n_1^4 - 4n_2^3m_2)HM(G_2) + [n_2^4 - 4n_2^3m_2]HM(G_1) + 20n_1M_1(G_1)F(G_2)
+20n_2M_1(G_2)F(G_1) + 10n_2^2m_2n_1M_1(G_1) + 10n_2n_1^2m_1M_1(G_2)
+8n_2^3m_2m_1 - 16n_2^2m_1^2M_1(G_2) - 16n_1m_2^2M_1(G_1) - 8n_1m_1^2F(G_2)
-8n_2^3m_2F(G_1) - 16n_1m_2^2M_1(G_2) - 16n_2^2m_2M_1(G_2) + 32m_1M_2(G_2)
+32m_2M_2(G_1)M_1(G_2) + 16n_2M_2(G_1)M_1(G_2) + 16n_1M_2(G_2)M_1(G_1) - 16F(G_1)F(G_2) - 32M_1(G_1)M_2(G_2)].

Proof. By the similar method in Proposition 3.7.

4. Application

Corollary 4.1. The hyper-Zagreb index of complement nanotube Figure 1 is given by

\[ HM(TiO_2[n, m]) = 12n(6mn + 6n - 1)^3[6m^2n + 12mn + 6n - 11m - 9]
+2680m^2n^2 + 4360mn^2 + 1696n^2 - 1036mn - 572n. \]
and since $M_1(\text{TiO}_2[n,m]) = 76mn + 48n$, given in [9]. $HM(\text{TiO}_2) = 580mn + 284n$, given in [12]. The partitions of the vertex set and edge set $V(\text{TiO}_2), E(\text{TiO}_2)$, of $\text{TiO}_2[n,m]$ nanotubes are given in Table 1. and Table 2., respectively. We have

$$HM(\overline{\text{TiO}_2[n,m]})$$
$$= 2\sum |V(\text{TiO}_2[n,m])|(|\sum |V(\text{TiO}_2[n,m])| - 1)^3$$
$$-12 \bigcup E(\text{TiO}_2[n,m])(|\sum |V(\text{TiO}_2[n,m])| - 1)^2 + 4 \bigcup E(\text{TiO}_2[n,m])^2$$
$$+(5\sum |V(\text{TiO}_2[n,m])| - 6)M_1(\text{TiO}_2[n,m]) - HM(\text{TiO}_2[n,m])$$
$$= 2(6mn + 6n)(6mn + 6n - 1)^3 - 12|E_{8}^*| + |E_{10}^* \cup E_{12}^*|$$
$$+|E_{15}^*|[6mn + 6n - 1]^2 + 4(|E_{8}^*| + |E_{10}^* \cup E_{12}^*| + |E_{15}^*|)^2$$
$$+(5(6mn + 6n)| - 6)(76mn + 48n) - 580mn - 284n$$
$$= 12n(6mn + 6n - 1)^2[6m^2n + 12mn + 6n - 11m - 9]$$
$$+2680m^2n^2 + 4360mn^2 + 1696n^2 - 1036mn - 572n.$$

Table 1. The vertex partition of $\text{TiO}_2[n,m]$ nanotubes.

<table>
<thead>
<tr>
<th>Vertex partition</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality</td>
<td>$2mn + 4n$</td>
<td>$2mn$</td>
<td>$2n$</td>
<td>$2mn$</td>
</tr>
</tbody>
</table>

Table 2. The edge partition of $\text{TiO}_2[n,m]$ nanotubes.

<table>
<thead>
<tr>
<th>Edge partition</th>
<th>$E_6 = E_8^*$</th>
<th>$E_7 = E_{10}^* \cup E_{12}^*$</th>
<th>$E_9 = E_{15}^*$</th>
<th>$E_{12}^*$</th>
<th>$E_{10}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality</td>
<td>$6n$</td>
<td>$4mn + 4n$</td>
<td>$6mn - 2n$</td>
<td>$4mn + 2n$</td>
<td>$2n$</td>
</tr>
</tbody>
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Figure 1. The molecular graph of $\text{TiO}_2[n,m]$ nanotube.
Corollary 4.2. Let $T = T[p,q]$ be the molecular graph of a nanotorus such that $|V(T)| = pq$, $|E(T)| = \frac{3}{2}pq$, Fig. 2. Then:


Proof. To proof (a), by using Theorem 2.1 we have

$$HM(T[p,q]) = 2|V(T[p,q])|(|V(T[p,q]) - 1)^3 - 12|E(T[p,q])|(|V(T[p,q]) - 1)^2$$

$$+ 4|E(T[p,q])|^2 + (5|V(T[p,q]) - 6)M_1(T[p,q]) - HM(T[p,q])$$.

And since $HM(T[p,q]) = 54pq$ by [8]. $M_1(T) = 9pq$ by [10]. Then

$$HM(T[p,q]) = pq[(pq - 1)^2[2pq - 20] + 54pq - 108].$$

To proof (b), by [8]. $HM(P_n \times T) = 250npq - 186pq$, $M_1(T) = 9pq$. $M_1(P_n \times T) = pq(25n - 18)$, and by using Lemma 2.1 $|E(P_n \times T)| = (n-1)pq + \frac{3}{2}npq = pq(\frac{3}{2}n - 1)$, $|V(P_n \times T)| = npq$ and by using Theorem 2.1 we get

$$HM(P_n \times T)$$

$$= 2|V(P_n \times T)|(|V(P_n \times T) - 1)^3 - 12|E(P_n \times T)|(|V(P_n \times T) - 1)^2$$

$$+ 4|E(P_n \times T)|^2 + (5|V(P_n \times T) - 6)M_1(P_n \times T) - HM(P_n \times T)$$


Figure 2. Molecular graph of a nanotorus
5. Conclusion

The present study has investigated some of the basic mathematical properties of the Hyper-Zagreb index of complement graphs and obtained explicit formula for their values under several graph operations. and we have studied the Hyper-Zagreb index of molecular complement graph of nanotorus and titania nanotubes $TiO_2[n,m]$.

References

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