ON SOME FUNCTIONS OF FAST INCREASE

K. SANTOSH REDDY and M. RANGAMMA

ABSTRACT. This article looks at some theorems of functions which satisfy the condition \( \lim_{\nu \to \infty} \frac{\ln \nu}{\ln \phi(\nu)} = 0 \). This function is labeled function of fast increase. To show the applicability of such functions, some general results on a sequence \( s_n \) of positive integers that satisfy the asymptotic rule \( s_n \sim n \ln \phi(n) \) where \( \phi(n) \) is a function of fast increase are derived.

1. INTRODUCTION

Drawing inspiration from functions of slow increase in [1, 2] the function of fast increase is defined as follows.

**Definition 1.1.** Let \( \phi(\nu) \) be a function from \([a, \infty)\) (where \( a > 0 \)) to \((0, \infty)\) and \( \phi(\nu) > 0 \) with continuous derivative \( \phi'(\nu) > 0 \) and \( \lim_{\nu \to \infty} \phi(\nu) = \infty \). The function \( \phi(\nu) \) is said to be function of fast increase if \( \lim_{\nu \to \infty} \frac{\ln \nu}{\ln \phi(\nu)} = 0 \). i.e., to every \( \sigma > 0 \) \( \exists k_\sigma \ni \nu > k_\sigma \) and \( \left| \frac{\ln \nu}{\ln \phi(\nu)} \right| < \sigma \)

\[ \Leftrightarrow |\ln \nu| < \sigma |\ln \phi(\nu)|, \ \forall \nu > k_\sigma \]

\[ \Leftrightarrow e^{\ln \nu} < e^{\sigma \ln \phi(\nu)}, \ \forall \nu > k_\sigma, \text{where } (\ln \nu, \ln \phi(\nu) > 0). \]

1 corresponding author

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Some functions of fast increase are \( \phi(\nu) = e^{\nu} \), \( \phi(\nu) = e^{e^{\nu}} \), \( \phi(\nu) = a^{\nu} \) (\( a \geq 2 \)), and \( \phi(\nu) = \frac{\Gamma(\nu)}{\Gamma'(\nu)} \) where \( \Gamma(\nu) = \int_0^{\infty} t^{\nu-1} e^{-t} dt \) etc.

**Note.** 1) If \( \phi(\nu) \) is function of fast increase, then \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu \phi'(\nu)} = 0 \).

2) Write \( F = \{ \phi| \phi \text{ is f.f.i.} \} \).

**Theorem 1.1.** Let \( \phi, \psi \in F \) and let \( \alpha, d > 0 \) be two constants, then \( \phi + d, \phi - d, d\phi, \phi\psi, \phi^{\alpha}, \phi \circ \psi, e^\phi \text{ and } \phi + \psi \) all lies in \( F \).

The proof is immediate consequence from the definition.

**Theorem 1.2.** Let \( \phi, \psi \in F \) and \( \mu(\nu) = \phi(\nu^{\alpha}) \) and \( \eta(\nu) = \phi(\nu^{\alpha}\psi(\nu)) \) for each \( \nu \) and \( \alpha > 1 \), then \( \mu, \eta \in F \).

**Proof.** (i) As \( \mu \) satisfies the conditions of a function of fast increase, we have
\[
\lim_{\nu \to \infty} \frac{\mu(\nu)}{\nu \mu'(\nu)} = \lim_{\nu \to \infty} \frac{\phi(\nu^{\alpha})}{\nu \phi'(\nu^{\alpha})} = \frac{\phi(\nu^{\alpha})}{\nu \phi'(\nu^{\alpha})},
\]
(\( \text{since } \nu \to \infty \text{ then } \nu^{\alpha} \to \infty \))

Therefore \( \mu = \phi(\nu^{\alpha}) \in F \).

(ii) As \( \mu \) satisfies the conditions of a function of fast increase, we have
\[
0 \leq \lim_{\nu \to \infty} \frac{\eta(\nu)}{\nu \eta'(\nu)} = \lim_{\nu \to \infty} \frac{\phi(\nu^{\alpha}\psi(\nu))}{\nu \phi'(\nu^{\alpha}\psi(\nu))} \left[ \alpha \nu^{\alpha-1} \psi(\nu) + \nu^{\alpha} \psi'(\nu) \right] \leq \lim_{\nu \to \infty} \frac{\phi(\nu^{\alpha}\psi(\nu))}{\nu \phi'(\nu^{\alpha}\psi(\nu))},
\]
(\( \text{since } \nu \to \infty \text{ and } \psi(\nu) \to \infty \text{ then } \nu^{\alpha} \psi(\nu) \to \infty \))

Therefore \( \eta = \phi(\nu^{\alpha}\psi(\nu)) \in F \). \( \square \)

**Theorem 1.3.** Let \( \phi, \psi \in F \) such that \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\psi(\nu)} = \infty \) and if \( \frac{d}{d\nu} \left( \ln \frac{\phi(\nu)}{\psi(\nu)} \right) > 0 \), then \( \frac{\phi}{\psi} \in F \).

**Proof.** Let \( \mu = \frac{\phi(\nu)}{\psi(\nu)} \) where \( \phi, \psi \in F \) and \( \mu' = \frac{\phi'\psi - \phi \psi'}{\psi^2} \). Then
\[
\lim_{\nu \to \infty} \frac{\mu}{\nu} = \lim_{\nu \to \infty} \frac{\phi(\nu)/\psi(\nu)}{\nu (\phi'(\nu)/\psi(\nu) - \phi(\nu)/\psi(\nu))} = \lim_{\nu \to \infty} \frac{1}{\nu} \left( \frac{\phi'(\nu)/\psi(\nu) - \phi(\nu)/\psi(\nu)}{\phi(\nu)/\psi(\nu)} \right) = 0
\]
\[
\left( \frac{d}{dx} \ln \frac{\phi(\nu)}{\psi(\nu)} \right) = \frac{\phi'(\nu)/\psi(\nu) - \phi(\nu)/\psi(\nu)}{\phi(\nu)/\psi(\nu)} > 0.
\]
Therefore \( \mu = \frac{\phi}{\psi} \in F. \)

**Theorem 1.4.** Let \( \phi(\nu) \) be a function from \([a, \infty) (a > 0)\) to \((0, \infty)\) such that \( \phi(\nu) > 0 \) with continuous derivative \( \phi'(\nu) > 0 \) and \( \lim_{\nu \to \infty} \phi(\nu) = \infty. \)

(i) Define \( \mu(\nu) = \phi(\ln \nu) \), then \( \mu \in F \iff \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu \phi'(\nu)} = 0 \).

(ii) Define \( \eta(\nu) = e^{\phi(\nu)} \), then \( \eta \in F \iff \lim_{\nu \to \infty} \frac{1}{\nu \phi'(\nu)} = 0 \).

**Proof.** (i) Suppose \( \mu \in F \Rightarrow \lim_{\nu \to \infty} \frac{\mu(\nu)}{\nu \mu'(\nu)} = 0 \Rightarrow \lim_{\nu \to \infty} \frac{\phi(\ln \nu)}{\nu \phi'/(\ln \nu)} = 0. \) If \( y = \ln \nu \) and \( \nu \to \infty \) then \( y \to \infty \) then \( \lim_{y \to \infty} \frac{\phi(y)}{\phi'(y)} = 0 \) or \( \lim_{\nu \to \infty} \frac{1}{\nu \phi'(\nu)} = 0. \)

Converse follows from the above proof.

(ii) Suppose \( \eta \in F \Rightarrow \lim_{\nu \to \infty} \frac{\eta(\nu)}{\nu \eta'(\nu)} = 0 \Rightarrow \lim_{\nu \to \infty} \frac{1}{\nu \phi'(\nu)} = 0 \).

Converse follows from the above proof. \( \square \)

**Theorem 1.5.** Let \( \phi \) be the function of fast increase if and only if to every \( \alpha > 0 \) then there exist \( \nu_\alpha \) such that \( \frac{d}{d\nu} \left[ \frac{\phi(\nu)}{\nu^\alpha} \right] > 0, \forall \nu > \nu_\alpha. \)

**Proof.** We have

\[
\frac{d}{d\nu} \left[ \frac{\phi(\nu)}{\nu^\alpha} \right] = \frac{\alpha \phi'(\nu) \nu^\alpha - \phi(\nu)}{\nu^{\alpha + 1}}.
\]

Suppose \( \phi \in F. \) Then \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu \phi'(\nu)} = 0, \) i.e. for every \( \alpha > 0 \) \( \exists \nu_\alpha : \forall \nu > \nu_\alpha \) and

\[
\left| \frac{\phi}{\nu \phi'} \right| < \frac{1}{\alpha}, \forall \nu > \nu_\alpha
\]

\[\Rightarrow \frac{1}{\alpha} - \frac{\phi}{\nu \phi'} > 0, \forall \nu > \nu_\alpha\]

\[\Rightarrow \frac{d}{d\nu} \left[ \frac{\phi(\nu)}{\nu^\alpha} \right] > 0, \forall \nu > \nu_\alpha,\]

from (1.1). Converse follows from the above proof. \( \square \)

**Theorem 1.6.** If \( \phi \in F \) then \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu^\beta} = \infty, \forall \beta > 0. \)

**Proof.** follows from the definition.

**Note.** We know that every \( \phi \in F \) is an increasing function. Moreover by the above theorem it is clear that \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu^\beta} = \infty, \forall \beta > 0. \)

This shows that the increasing nature of \( \phi \) is fast.
Corollary 1.1. If \( \phi \in F \) then \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\nu} = \infty \) and \( \lim_{\nu \to \infty} \phi'(\nu) = \infty \).

The proof is immediate consequence of Theorem 1.6.

Theorem 1.7. If \( \phi \in F \), for any \( \alpha \geq -1 \) and \( \beta \in R^+ \), the series \( \sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta \) diverges to \( \infty \).

Proof. Write \( \sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta = \sum_{j=1}^{\infty} \left[ j^{\alpha+1} \phi(j)^\beta \right] \frac{1}{j} \). We know that \( \sum_{j=1}^{\infty} \frac{1}{j} \) diverges to \( \infty \).

Given \( \alpha \geq -1 \Rightarrow \alpha + 1 \geq 0 \) and \( \beta \in R^+ \), \( \Rightarrow \lim_{j \to \infty} j^\alpha \phi(j)^\beta = \infty \).

Therefore \( \sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta \) diverges to \( \infty \). \( \square \)

Theorem 1.8. Let \( \phi \in F \), for any \( \alpha \geq -1 \) and \( \beta \in R^+ \) Then

\[
\lim_{v \to \infty} \frac{\int_{a}^{v} x^\alpha \phi(x)^{\beta-1} \phi'(x) dx}{\frac{1}{\beta} v^\alpha \phi(v)^{\beta}} = 1.
\]

Proof. From Theorem 1.7., \( \lim_{v \to \infty} \frac{1}{\beta} v^\alpha \phi(v)^{\beta} = \infty \), \( \alpha \geq -1 \), \( \beta \in R^+ \) and

\[
\sum_{v=1}^{\infty} v^\alpha \phi(v)^{\beta} = \infty
\]

\( \Rightarrow \lim_{v \to \infty} \int_{a}^{v} x^\alpha \phi(x)^{\beta-1} \phi'(x) dx = \infty
\]

\[
\Rightarrow \lim_{v \to \infty} \frac{\int_{a}^{v} x^\alpha \phi(x)^{\beta-1} \phi'(x) dx}{\frac{1}{\beta} v^\alpha \phi(v)^{\beta}} = \lim_{v \to \infty} \frac{v^\alpha \phi(v)^{\beta-1} \phi'(v)}{\nu^\alpha \phi(\nu)^{\beta-1} \phi'(\nu)} \left[ \frac{\phi(\nu)}{\nu^\beta \phi'(\nu)} + 1 \right] = 1,
\]

by L’Hospital’s Rule and \( \phi \in F \). \( \square \)

Corollary 1.2. If \( \phi \in F \), then the following results holds.

(i) \( \int_{a}^{v} x^\alpha \phi'(x) dx \sim v^\alpha \phi(v) \).

(ii) \( \int_{a}^{v} \frac{\phi'(x)}{x} dx \sim \frac{\phi(v)}{\nu} \).

Proof is a particular case of Theorem 1.8.

Theorem 1.9. If \( \phi \in F \) and \( \lim_{\nu \to \infty} \frac{\phi(\nu)}{\phi'(\nu)} = M \) then \( \lim_{\nu \to \infty} \frac{1}{n\phi(\nu + k)} = 1 \) for every \( k \in R \).
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Proof. Let \( \psi(\nu) = \ln \phi(\nu) \), Suppose \( k > 0 \), \( \exists x \in (\nu, \nu + k) \) such that
\[
\psi(\nu + k) - \psi(\nu) = (\nu + k - \nu) \psi'(x) \quad \text{(By LMVT)}
\]
\[
\Rightarrow \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = \frac{k\psi'(x)}{\psi(\nu)}
\]
\[
\Rightarrow \lim_{\nu \to \infty} \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = k \lim_{\nu \to \infty} \left( \frac{\phi'(\nu)}{\phi(\nu)} \times \frac{1}{1n\phi(\nu)} \right) = 0,
\]
\[
\Rightarrow \lim_{\nu \to \infty} \frac{1n\phi(\nu + k)}{1n\phi(\nu)} = 1.
\]
Suppose \( k < 0 \), \( \exists x \in (\nu + k, \nu) \) such that
\[
\psi(\nu) - \psi(\nu + k) = (\nu - \nu - k) \psi'(x) \quad \text{(By LMVT)}
\]
\[
\Rightarrow \lim_{\nu \to \infty} \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = -k \lim_{\nu \to \infty} \left( \frac{\phi'(\nu)}{\phi(\nu)} \times \frac{1}{1n\phi(\nu)} \right) = 0,
\]
\[
\Rightarrow \lim_{\nu \to \infty} \frac{1n\phi(\nu + k)}{1n\phi(\nu)} = 1.
\]
\[\square\]

2. APPLICATIONS OF FUNCTIONS OF FAST INCREASING TO SOME SEQUENCES OF INTEGERS

Definition 2.1. Let \( \phi \in F \), \((s_n)\) be a sequence of positive integers and is strictly increasing such that \( s_1 \geq 1 \) and

\[
\lim_{n \to \infty} \frac{s_n}{n^{r} \ln \phi(n)} = 1 \quad \text{for some } r \geq 1
\]

i.e. \( s_n \sim n^r \ln \phi(n) \), see reference [3].

For example, (i) \((s_n) = n^2, \phi(\nu) = e^\nu \) and \( r = 1 \)
(ii) \((s_n) = n e^n, \phi(\nu) = e^{e^\nu} \) and \( r = 1 \)

Definition 2.2. Let \((s_n)\) be the sequence defined as above and \( \nu > 0 \) then \( f(\nu) = \sum_{s_k \leq \nu} 1 \), i.e. the member of \((s_n)\) and is not exceeding \( \nu \).

Theorem 2.1. Let \((s_n)\) be the sequence satisfying and \( \phi, \psi \in F \) then

(i) \( s_{n+1} \sim s_n \)
(ii) \( \lim_{n \to \infty} \frac{s_{n+1} - s_n}{s_n} = 0 \)
(iii) \( \ln s_{n+1} \sim \ln s_n \)
(iv) $\psi(s_{n+1}) \sim \psi(s_n)$

(v) $\lim_{\nu \to \infty} \frac{f(\nu)}{\nu} = 0$.

Proof. The proofs are immediate consequence of (2.1) and Theorem 1.9. □

Theorem 2.2. Let $(s_n)$ be the sequence satisfying 2.1 and $\psi \in F$ and $p \geq 1$ then

$$\psi(s_n) \sim p \psi(n) \iff \psi(f(\nu)) \sim \frac{1}{p} \psi(\nu).$$

Proof. Let $\psi(f(\nu)) \sim \frac{1}{p} \psi(\nu)$

$$\Rightarrow \psi(f(s_n)) \sim \frac{1}{p} \psi(s_n) \Rightarrow \psi(s_n) \sim p \psi(n) \quad (f(s_n) = n).$$

Conversely, let

$$(2.2) \quad \psi(s_n) \sim p \psi(n) \Rightarrow \lim_{n \to \infty} \frac{\psi(f(s_n))}{p \psi(s_n)} = 1 \quad \text{ (since } f(s_n) = n).$$

If $s_n \leq \nu \leq s_{n+1}$ then $\psi(f(s_n)) \leq \psi(f(\nu)) \leq \psi(f(s_{n+1}))$, and

$$\frac{1}{p} \psi(s_n) \leq \frac{1}{p} \psi(\nu) \leq \frac{1}{p} \psi(s_{n+1}).$$

$$\Rightarrow \lim_{n \to \infty} \frac{\psi(f(s_n))}{p \psi(s_n)} \leq \lim_{\nu \to \infty} \frac{\psi(f(\nu))}{p \psi(\nu)} \leq \lim_{n \to \infty} \frac{\psi(f(s_{n+1}))}{p \psi(s_n)}$$

$$\Rightarrow 1 \leq \frac{\psi(f(\nu))}{\frac{1}{p} \psi(\nu)} \leq 1 \quad (s_{n+1} \sim s_n)$$

and from (2.2),

$$\Rightarrow \psi(f(\nu)) \sim \frac{1}{p} \psi(\nu).$$

□

Lemma 2.1. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms and $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. If $\sum_{n=1}^{\infty} b_n$ is divergent then $\sum_{k=1}^{n} \frac{a_k}{b_k} = 1$.

Theorem 2.3. Let $(s_n)$ be the sequence satisfying (2.1) $\psi \in F$ space, $p \geq 1$, $\psi(s_n) \sim \psi(n)$ and $\psi'(s_n) \sim \psi'(n)$ then

$$s_n \sim \frac{1}{n} \psi(n) \iff f(\nu) \sim \frac{\psi(\nu)}{\nu} \iff f(\nu) \sim \int_{a}^{\nu} \frac{\psi'(x)}{x} \, dx \iff \sum_{s_k \leq \nu} k \psi'(s_k) \sim \frac{\psi(\nu)^2}{\nu}.$$
Proof. Suppose \( f(\nu) \sim \frac{\psi(\nu)}{\nu} \Rightarrow f(s_n) \sim \frac{\psi(s_n)}{s_n}, \)

\[ \Rightarrow n \sim \frac{\psi(s_n)}{s_n} \Rightarrow s_n \sim \frac{1}{n} \psi(n) \quad \text{(Since } f(s_n) = n). \]

Conversely, suppose

\[ s_n \sim \frac{1}{n} \psi(n) \Rightarrow \lim_{n \to \infty} \frac{f(s_n)}{\psi(s_n)} = 1. \]

If \( s_n \leq \nu \leq s_{n+1} \) then \( f(s_n) \leq f(\nu) \leq f(s_{n+1}) \), and \( \frac{\psi(s_n)}{s_n} \leq \frac{\psi(\nu)}{\nu} \leq \frac{\psi(s_{n+1})}{s_{n+1}} \).

\[ \Rightarrow \lim_{n \to \infty} \frac{f(s_n)}{\psi(s_n)} \leq \lim_{\nu \to \infty} \frac{f(\nu)}{\psi(\nu)} \leq \lim_{n \to \infty} \frac{f(s_{n+1})}{\psi(s_{n+1})}. \]

\[ \Rightarrow 1 \leq \lim_{\nu \to \infty} \frac{f(\nu)}{\psi(\nu)} \leq 1 (s_{n+1} \sim s_n) \Rightarrow f(\nu) \sim \frac{\psi(\nu)}{\nu}. \]

We have

\[ \int_a^\nu \frac{\psi'(x)}{x} \, dx \sim \frac{\psi(\nu)}{\nu} \Rightarrow f(\nu) \sim \int_a^\nu \frac{\psi'(x)}{x} \, dx. \]

Also we have \( \int_a^\nu x \psi'(x) \, dx \sim \nu \psi(\nu) \) and \( \psi'(x) \) is increasing

\[ \Rightarrow \sum_{k=1}^n k \psi'(k) = \int_a^\nu x \psi'(x) \, dx + h(n). \]

\[ \Rightarrow \int_a^\nu x \psi'(x) \, dx + h(n) \sim \nu \psi(\nu). \]

From Lemma 2.1., we can write \( \sum_{k=1}^n k \psi'(s_k) \sim \sum_{k=1}^n k \psi'(k) \)

\[ \Rightarrow \sum_{k=1}^n k \psi'(s_k) \sim n \psi(n) \]

\[ \Rightarrow \sum_{s_k \leq s_n} k \psi'(s_k) \sim n \psi(s_n) \]

\[ \Rightarrow \sum_{s_k \leq s_n} k \psi'(s_k) \sim f(s_n) \psi(s_n) \]

\[ \Rightarrow \lim_{n \to \infty} \frac{f(s_n)}{\left( \sum_{s_k \leq s_n} k \psi'(s_k) \right) / \psi(s_n)} = 1. \quad (2.3) \]
If \( s_n \leq \nu \leq s_{n+1} \) then \( f(s_n) \leq f(\nu) \leq f(s_{n+1}) \), and
\[
\sum_{s_k \leq s_n} k\psi'(s_k) \leq \sum_{s_k \leq \nu} k\psi'(s_k) \leq \sum_{s_k \leq s_{n+1}} k\psi'(s_k),
\]
\[
\Rightarrow \lim_{n \to \infty} \frac{f(s_n)}{\psi(s_n)} \leq \lim_{\nu \to \infty} \frac{f(\nu)}{\psi(\nu)} \leq \lim_{n \to \infty} \frac{f(s_{n+1})}{\psi(s_{n+1})}.
\]
\[
\Rightarrow 1 \leq \lim_{\nu \to \infty} \frac{f(\nu)}{\psi(\nu)} \leq 1 \quad \text{(since \( s_{n+1} \sim s_n \)) and from (2.3)}
\]
\[
\Rightarrow f(\nu) \sim \frac{\sum_{s_k \leq \nu} k\psi'(s_k)}{\psi(\nu)}.
\]
\[
\Rightarrow f(\nu) \sim \psi(\nu) \frac{\sum_{s_k \leq \nu} k\psi'(s_k)}{\psi(\nu)}.
\]
\[
\Rightarrow \sum_{s_k \leq \nu} k\psi'(s_k) \sim \psi(\nu)^2.
\]

\[\square\]

**Theorem 2.4.** Let \((s_n)\) be the sequence satisfying (2.1), \( \psi \in F, p \geq 1, l \geq 1, \psi(s_n) \sim l\psi(n) \) and \( \psi'(s_n) \sim l\psi'(n) \) then
\[
s_n \sim \frac{1}{np} \psi(n) \iff f(\nu) \sim \frac{\psi(\nu)^{1/p}}{l^{1/p}p^{1/p}} \iff f(\nu) \sim \frac{1}{lp^{1/p}} \int_a^\nu \psi(x)^{\frac{1}{p}-1} \psi'(x) \, dx
\]
\[
\iff \sum_{s_k \leq \nu} k\psi'(s_k) \sim \psi(\nu)^{\frac{1}{p}+1}.
\]

**Proof.** Suppose \( f(\nu) \sim \frac{\psi(\nu)^{1/p}}{l^{1/p}p^{1/p}} \Rightarrow f(s_n) \sim \frac{\psi(s_n)^{1/p}}{l^{1/p}s_n^{1/p}} \)
\[
\Rightarrow n \sim \frac{\psi(s_n)^{1/p}}{l^{1/p}s_n^{1/p}} \quad \text{(Since \( f(s_n) = n \)}
\]
\[
\Rightarrow s_n \sim \frac{1}{np^p} \psi(n) \quad \text{(Since \( \psi(s_n) \sim l\psi(n) \).}
\]

Conversely, suppose \( s_n \sim \frac{1}{np^p} \psi(n) \)
\[
(2.4) \Rightarrow \lim_{n \to \infty} \frac{f(s_n)}{\psi(s_n)^{1/p}} = 1.
\]
If \( s_n \leq \nu \leq s_{n+1} \) then \( f(s_n) \leq f(\nu) \leq f(s_{n+1}) \), and \( \frac{\psi(s_n)}{s_n} \leq \frac{\psi(\nu)}{\nu} \leq \frac{\psi(s_{n+1})}{s_{n+1}} \).

\[
\lim_{n \to \infty} \frac{f(s_n)}{(\psi(s_n))^{1/p}} \leq \lim_{\nu \to \infty} \frac{f(\nu)}{(\psi(\nu))^{1/p}} \leq \lim_{n \to \infty} \frac{f(s_{n+1})}{(\psi(s_{n+1}))^{1/p}}
\]

\[
1 \leq \lim_{\nu \to \infty} \frac{f(\nu)}{(\psi(\nu))^{1/p}} \leq 1 \text{ (Since } s_{n+1} \sim s_n) \]

and from (2.4)

\[
\Rightarrow f(\nu) \sim \frac{\psi(\nu)^{1/p}}{1/p}.
\]

We have

\[
\int_a^\nu x^\alpha \psi(x)^{\beta-1} \psi'(x)dx \sim \frac{1}{\beta} x^\alpha \psi(x)^\beta.
\]

By taking \( \alpha = \frac{1}{p}, \beta = -\frac{1}{p} \) then we get

\[
f(\nu) \sim \frac{1}{p} \int_a^\nu \psi(x)^{\frac{1}{p}-1} \psi'(x)dx.
\]

The proof that \( \sum_{k \leq \nu} k \psi'(s_k) \sim \frac{\psi(\nu)^{p+1}}{p^{1/p} \nu^{1/p}} \) is the same as in Theorem 2.3. \( \square \)

3. Conclusions

The results discussed in this article are employed in examples to show their applicability in number theory.

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References

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF SCIENCE
OSMANIA UNIVERSITY
HYDERABAD -500007, TELANGANA, INDIA
AND
DEPARTMENT OF HUMANITIES AND SCIENCES
VARDHAMAN COLLEGE OF ENGINEERING
SHAMSHABAD,HYDERABAD-501 218, TELANGANA, INDIA
E-mail address: santureddyk@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF SCIENCE
OSMANIA UNIVERSITY
HYDERABAD -500007, TELANGANA, INDIA
E-mail address: rangamma1999@gmail.com