1. INTRODUCTION

Ramanujan’s theta function \( f(x,y) \) is defined as
\[
f(x,y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2} \quad |xy| < 1.
\]
The function \( f(x,y) \) enjoys the well-known Jacobi’s triple-product identity [5, p. 35] given by
\[
f(x,y) = (−x; xy)_\infty (−y; xy)_\infty (xy; xy)_\infty,
\]
where here and throughout the paper, we assume \(|q| < 1\) and employ the standard notation
\[
(x; q)_\infty := \prod_{n=0}^{\infty} (1 - xq^n).
\]
The important special cases of \( f(x, y) \) [5, p. 36] are as follows:

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \left( \frac{q^2}{q^2} \right)_\infty, \\
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \\
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n-1)/2} = (q; q)_\infty.
\]

After Ramanujan, we define 
\[
\chi(q) := (-q; q^2)_\infty.
\]

For convenience we write \( f(-q^n) = f_n \). Also, one can easily see that

\[
\varphi(q) = \frac{f_2}{f_1 f_4}, \quad \chi(q) = \frac{f_2}{f_1 f_4} \quad \text{and} \quad \chi(-q) = \frac{f_1}{f_2}.
\]

A theta function identity which relates \( f_1, f_2, f_n \) and \( f_{2n} \) is called a theta function identity of level \( 2n \). Ramanujan documented many theta functions which involve quotients of the function \( f_1 \) at different arguments. For example, if \( 6, \text{ p. 206} \)

\[
P := \frac{f_1}{q^{1/6} f_5} \quad \text{and} \quad Q := \frac{f_2}{q^{1/3} f_{10}}
\]

then

\[
PQ + \frac{5}{PQ} = \left( \frac{Q}{P} \right)^3 + \left( \frac{P}{Q} \right)^3.
\]

B. C. Berndt [5] proved similar type of identities and used it to evaluate various continued fractions, weber class invariants, theta functions and many more. After the publication of [5,6], many mathematicians discovered similar identities in the spirit of Ramanujan. For the wonderful work, one can see [1–4, 9, 16, 17]. Motivated by the above work, M. Somos [11] used a computer to discover around 6277 new elegant Dedekind eta-function identities of various levels without offering the proof. He runs PARI/GP scripts and it works as a sophisticated programmable calculator. Many authors [12–15, 18] have given the proof of Somos’s identities of various levels and found the applications of these in colored partitions. S. Cooper [7, 8] proved some Dedekind eta-function identities of level 6 while finding series and iterations for \( 1/\pi \). These identities are also recorded by Somos [11] using PARI/GP scripts. Motivated by this, in the present work we prove these identities by using modular equation of degree 3 in Section 2. As an application of this, we obtain interesting combinatorial interpretations of colored
partitions in Section 3. Before that we define a modular equation as given in the literature. A modular equation of degree $n$ is an equation relating $\alpha$ and $\beta$ that is induced by

$$n \frac{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha \right)}{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)} = \frac{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \beta \right)}{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \beta \right)},$$

where

$$2F_1(p, q; r; x) := \sum_{n=0}^{\infty} \frac{(p)_n(q)_n}{(r)_n n!} x^n \quad |x| < 1,$$

denotes an ordinary hypergeometric function with 

$$(p)_n := p(p+1)(p+2)\ldots(p+n-1).$$

Then, we say that $\beta$ is of degree $n$ over $\alpha$ and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)$ and $z_n = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \beta \right)$.

2. MAIN RESULTS

**Theorem 2.1.** We have

$$\psi^2(q) - q\psi^2(q^5) = \frac{f_2f_3^3}{q^{1/4}f_{10}}.$$

**Proof.** The following modular equation of degree 5 is recorded by Ramanujan on page 236 of his second notebook [10] and Entry 13(ix) and (xiv) [5, pp. 280 – 288]:

$$1 + 4^{1/3} \left( \frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)} \right)^{1/12} = \frac{m}{2} \left( 1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \right),$$

$$1 + 4^{1/3} \left( \frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)} \right)^{1/12} = \frac{5}{2m} \left( 1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \right)$$

and if

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \quad \text{and} \quad Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/8}$$

then, we have

$$Q + \frac{1}{Q} + 2 \left( P - \frac{1}{P} \right) = 0,$$
where $\beta$ has degree 5 over $\alpha$ and $m$ is the multiplier of degree 5. From (2.1) and (2.2), we have

$$
(2.4) \quad \frac{m^2}{5} = \frac{1 + 4^{1/3} \left( \frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)} \right)^{1/12}}{1 + 4^{1/3} \left( \frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)} \right)^{1/12}}.
$$

From Entry 10(i) and 12(v) [5, pp. 122–124], we have

$$
(2.5) \quad \varphi(q) := \sqrt{z}
$$

and

$$
(2.6) \quad \chi(q) := 2^{1/6} \left( \frac{x(1-x)}{q} \right)^{-1/24}
$$

where $q = e^y$ and $y = \pi \, _2F_1 \left( 1/2, 1/2; 1; 1-x \right) / _2F_1 \left( 1/2, 1/2; 1; x \right)$. From (2.6), we can write

$$
\chi(q) = 2^{1/6} \left( \frac{q}{\alpha(1-\alpha)} \right)^{1/24} \quad \text{and} \quad \chi(q^5) = 2^{1/6} \left( \frac{q^5}{\beta(1-\beta)} \right)^{1/24}.
$$

From the above, we deduce

$$
2^{4/3} q^2 \frac{\chi^2(q)}{\chi^{10}(q^5)} = \left( \frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)} \right)^{1/12} \quad \text{and} \quad 2^{4/3} \frac{\chi^2(q^5)}{\chi^{10}(q)} = \left( \frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)} \right)^{1/12}.
$$

On using (2.5) and the above, in (2.4), we obtain

$$
(2.7) \quad \frac{\varphi^4(q)}{5 \varphi^4(q^5)} = \frac{1 + 4^{2/3} \frac{\chi^2(q)}{\chi^{10}(q^5)}}{1 + 4^{2/3} \frac{\chi^2(q^5)}{\chi^{10}(q)}}.
$$

Similarly, on transcribing (2.3) into theta function, we obtain

$$
(2.8) \quad \frac{u^3}{v^3} + \frac{v^3}{u^3} + \frac{4}{u^3 v^3} - u^3 v^3 = 0,
$$

where,

$$
u := u(q) = q^{-1/24} \chi(q) \quad \text{and} \quad v := v(q) = q^{-5/24} \chi(q^5).
$$

On multiplying (2.8) throughout by $(uv)^{-10}(u^6 + 4uv + u^5 v^5 - v^6)$, we obtain

$$
\frac{v^2}{u^{10}} + \frac{16}{v^8 u^8} - 1 - \frac{2u}{v^5} - \frac{8}{u^9 v^3} - \frac{u^2}{v^{10}} = 0,
$$

which is equivalent to

$$
\frac{5u^2}{u^{10}} \left( 1 + \frac{4u^2}{v^{10}} \right) = \left( 1 + \frac{u}{v^5} \right)^{2} \left( 1 + \frac{4v^2}{u^{10}} \right).
$$
Employing (2.7) in the above, we see that
\begin{equation}
\frac{v}{u^5} \frac{\varphi^2(q)}{\varphi(q^5)} = 1 + \frac{u}{v^5}.
\end{equation}

From (1.1), we observe
\begin{equation}
\frac{\varphi(q)}{\varphi(q^5)} = \frac{u^2}{q^{1/3}v^2 f_{10}}.
\end{equation}

Using (2.10) in (2.9), we have
\[
q^{-2/3} \left( \frac{f_2}{f_{10}} \right)^2 - \frac{u}{v^5} - 1 = 0.
\]

Letting \( q \to -q \) in the above, rewriting \( u(-q) \) and \( v(-q) \) in terms of \( f_n \) by employing (1.1) and then multiplying throughout by \( f_1^2 f_5^2 f_{10} \), we deduce the result. \( \square \)

**Theorem 2.2.** We have
\[
\varphi^2(-q^5) - \varphi^2(-q) = \frac{4q f_1 f_{10}^3}{f_2 f_5^2}.
\]

**Proof.** On multiplying (2.8), throughout by \( 4(uv)^{-10}(u^6 + 4uv + u^5v^5 - v^6) \), we obtain
\[
4 - \frac{4v^2}{u^{10}} + \frac{8u}{v^5} + \frac{32}{u^9v^3} + \frac{4u^2}{v^{10}} - \frac{64}{v^8u^8} = 0,
\]
which can be rewritten as
\[
\left( 1 - \frac{4u}{v^5} \right)^2 \left( 1 + \frac{4v^2}{u^{10}} \right) - 5 - \frac{20u^2}{v^{10}} = 0.
\]

Employing (2.7) in the above, we see that
\[
\left( \frac{v^4}{u^4} + \frac{4}{u^3v} \right) \frac{u^4 \varphi^2(q^5)}{v^2 \varphi^2(q)} = 1.
\]

Employing (2.10) in the above, we obtain
\[
\left( \frac{v^4}{u^4} + \frac{4}{u^3v} \right) \left( \frac{f_{10}}{f_2} \right)^2 = q^{-2/3}.
\]

Letting \( q \to -q \) in the above, rewriting \( u(-q) \) and \( v(-q) \) in terms of \( f_n \) by employing (1.1) and then multiplying throughout by \( f_1^2 f_5^2 f_{10} \), we deduce the result. \( \square \)

### 3. Application to Partitions

For simplicity, in sequel we employ the notation
\[
(x_1, x_2, \ldots, x_n; q)_\infty := (x_1; q)_\infty (x_2; q)_\infty \ldots (x_n; q)_\infty,
\]
and define,

\[(q^r; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \quad r, s \in \mathbb{N}; r < s.\]

For example of this, \((q^{3 \pm 1}; q^5)_\infty\) means \((q^3, q^5; q^8)_\infty\), which is \((q^5; q^8)_\infty\). Now we define colored partition as defined as in the literature.

“A positive integer \(n\) has \(l\) colors if there are \(l\) copies of \(n\) available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions”.

For instance if 3 colors are assigned to 1, then the possible colored partitions of 2 are \(1_b + 1_b, 1_y + 1_y, 1_i + 1_i, 1_b + 1_y, 1_b + 1_i, 1_i + 1_y\) and 2, where we utilized the indices \(b\) (blue), \(y\) (yellow) and \(i\) (indigo) to recognize three colors of 1. Also \(\frac{1}{(q^a; q^b)^k}_\infty\) is the generating function for the number of partitions of \(n\) where all the parts are congruent to \(a \pmod{b}\) and have \(k\) colors. In this section we demonstrate this by giving the combinational interpretations for our results discussed in Section 2.

**Theorem 3.1.** Let \(\alpha(n)\) represent the number of partitions of \(n\) being divided into parts congruent to \(\pm 1\) or \(\pm 3\) modulo 10 with two colors each and \(+5\) modulo 10 with five colors. Let \(\beta(n)\) indicate the number of partitions of \(n\) being split into parts congruent to \(\pm 2\) or \(\pm 4\) modulo 10 with two colors each. Let \(\gamma(n)\) taken to represent the number of partitions of \(n\) into several parts congruent to \(\pm 1, \pm 3\) or \(+5\) modulo 10 with one color each and \(\pm 2\) or \(\pm 4\) modulo 10 with two colors each, then we have

\[\alpha(n) - \beta(n) = \gamma(n) = 0, \quad n \geq 1.\]

**Proof.** On rewriting Theorem 2.1 using (1.1) and then dividing throughout by \(f_1^3 f_2^2 f_3^3 f_{10}^5\), we obtain

\[(3.1) \quad \frac{1}{f_1^2 f_3^4 f_{10}^2} - \frac{q}{f_2^4 f_{10}^2} - \frac{1}{f_1^3 f_2^2 f_{10}^5} = 0.\]

Rewriting the above subject to the common base \(q^{10}\), and using the known fact \((q^{a \pm a}; q^b)_\infty = (q^a, q^{b-a}; q^b)_\infty\) for \(a, b \in \mathbb{Z}^+\) with \(a < b\), (3.1) reduces to

\[\frac{1}{(q_{1 \pm 1}, q_{2 \pm 3}, q_{1 \pm 5}; q^{10})_\infty} - \frac{q}{(q_{2 \pm 2}, q_{2 \pm 4}; q^{10})_\infty} - \frac{1}{(q_{1 \pm 1}, q_{2 \pm 2}, q_{1 \pm 3}, q_{2 \pm 4}, q_{1 \pm 5}; q^{10})_\infty} = 0.\]

The above identity generates \(\alpha(n)\), \(\beta(n)\) and \(\gamma(n)\) as the generating functions and hence, we have

\[\sum_{n=0}^{\infty} \alpha(n)q^n - q \sum_{n=0}^{\infty} \beta(n)q^n - \sum_{n=0}^{\infty} \gamma(n)q^n = 0.\]
Now, on extracting the powers of $q^n$ in the above, we obtain the result. □

Following table verifies the partitions for $n = 2$.

<table>
<thead>
<tr>
<th>$\alpha(2)$</th>
<th>$1_r + 1_r, 1_y + 1_y, 1_y + 1_y, 1_r + 1_y$</th>
</tr>
</thead>
<tbody>
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<td>$\beta(1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\gamma(2)$</td>
<td>$1 + 1, 2_r, 2_m$</td>
</tr>
</tbody>
</table>

**Theorem 3.2.** Let $\alpha(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1$ or $\pm 3$ modulo 10 with four colors each, $\pm 2$ or $\pm 4$ modulo 10 with two colors each and $+5$ modulo 10 with one color. Let $\beta(n)$ represent the number of partitions of $n$ being divided into parts congruent to $+5$ modulo 10 with one color. Let $\gamma(n)$ indicate the number of partitions of $n$ being split into parts congruent to $\pm 1$ or $\pm 3$ modulo 10 with three colors each, $\pm 2$, or $\pm 4$ modulo 10 with two colors each, then we have

$$\alpha(n) - \beta(n) - 4\gamma(n - 1) = 0, \quad n \geq 1.$$

**Proof.** On rewriting Theorem 2.2 using (1.1) and then dividing throughout by $f_1^4f_2^2f_5^3f_{10}^5$, we obtain

$$\frac{1}{f_1^2f_5^3f_{10}^4} - \frac{1}{f_2^2f_{10}^5} - \frac{4q}{f_1^3f_5^4f_{10}^5} = 0.$$  

(3.2)

Rewriting the above subject to the common base $q^{10}$, and using the known fact $(q^{\pm a}, q^b)_\infty = (q^a, q^{b-a}, q^b)_\infty$ for $a, b \in \mathbb{Z}^+$ with $a < b$, (3.2) reduces to

$$\frac{1}{(q_1^{\pm 1}, q_2^{\pm 2}, q_3^{\pm 3}, q_4^{\pm 4}, q_5^{\pm 5}, q^{10})_\infty} - \frac{1}{(q_1^{+5}, q^{10})_\infty} - \frac{4q}{(q_3^{+1}, q_2^{+2}, q_3^{+3}, q_4^{+4}, q^{10})_\infty} = 0.$$

The above identity generates $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ as the generating functions and hence, we have

$$\sum_{n=0}^{\infty} \alpha(n)q^n - \sum_{n=0}^{\infty} \beta(n)q^n - q \sum_{n=0}^{\infty} \gamma(n)q^n = 0.$$

Now, on extracting the powers of $q^n$ in the above, we obtain the result. □

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<tr>
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