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A NINTH ORDER ITERATIVE METHOD FOR SOLVING NON-LINEAR EQUATIONS WITH HIGH-EFFICIENCY INDEX

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ABSTRACT. Establish a new ninth order iterative method to 7 solve non-linear equations. In this paper, a new scheme of a new modification of Newton's method with higher-order convergence is proposed and proved that this scheme is of ninth order of convergence. Using some numerical examples concluded a new ninth order scheme is better than 7 Newton's method and other defined methods with the same order.

1. INTRODUCTION

In applied mathematics, recently a lot of investigation going on approximate to find the root of the nonlinear equation

$$g\,(t) = 0 \tag{1.1}$$

where $f : D \to R$ is a scalar function, D is an open interval. Therefore, the design of the iterative scheme for solving non-linear scalar function is an interesting and important task in numerical analysis.

In this method of finding a zero of non-linear equations, Newton's method (NR) [2] is one of the optimal second- order method to obtain the root of (1.1)

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is given by

$$t_{n+1} = t_n - \frac{g(t_n)}{g'(t_n)} n = 0, 1, 2, \cdots$$

and the NR method converge quadratically and its efficiency index is $\sqrt{2} = 1.414$.

An iterative method with ninth order convergence (ZHONG) for solving non-linear equations proposed by Zhongyong et.al, [8] is given by

$$\begin{aligned} y_n &= t_n - \frac{g(t_n)}{g'(t_n)} \\ z_n &= y_n - \left\{1 + \left(\frac{g(y_n)}{g(t_n)}\right)^2\right\} \frac{g(y_n)}{g'(y_n)} \\ t_{n+1} &= z_n - \left\{1 + 2\left(\frac{g(y_n)}{g(t_n)}\right)^2 + 2\frac{g(z_n)}{g(y_n)}\right\} \frac{g(z_n)}{g'(y_n)}. \end{aligned}$$

A quadrature based three-step ninth-order iterative method (SK) proposed by Khattri [6] is given by

$$\begin{aligned} y_n &= t_n - \frac{g(t_n)}{g'(t_n)} \\ z_n &= y_n - \frac{(t_n - y_n)\, g(y_n)}{g(t_n) - 2g(y_n)} \\ x_{n+1} &= z_n - \frac{g(z_n)\, g'(z_n)}{(g'(z_n))^2 - g(z_n)\left[\frac{g(z_n) - g(t_n) - g'(t_n)(z_n - t_n)}{(z_n - t_n)^2}\right]}. \end{aligned}$$

New ninth order J-Halley method for solving non-linear equations (FA) proposed by Farooq et.al, [1] is given by

$$\begin{aligned} y_n &= t_n - \frac{2h(t_n)}{3h'(t_n)} \\ z_n &= t_n - J_f \frac{h(t_n)}{h'(t_n)} \\ x_{n+1} &= z_n - \frac{2h(z_n)\, h'(z_n)}{2h'(z_n) - h(z_n)\, L} \end{aligned}$$

where $J_f = \dfrac{3h'(y_n) + h'(t_n)}{6h'(y_n) - 2h'(t_n)}$ and $L = \dfrac{h'(z_n) - h'(t_n)}{z_n - t_n}$.

In section 2, we defined the new three-step iterative method and in section 3, we concluded our method is converging with order nine. Finally in section 4, using some defined examples we conformed that our new scheme is better than other methods with the same order of convergence.

2. NINTH ORDER CONVERGENT (MSK) METHOD

Consider t^* is an exact root of (1.1) where g(t) is continuous and has well defined first derivatives. Let t_n be the root of nth approximation of (1.1) and is

$$t^* = t_n + \varepsilon_n \,, \tag{2.1}$$

where ε_n is the error. Thus, we get

$$g(t^*) = 0\,. \tag{2.2}$$

Writing $g(t^*)$ by Taylor's series about t_n, we have $g(t^*) = g(t_n) + (t^* - t_n)\, g'(t_n) + \frac{(t^*-t_n)^2}{2!} g''(t_n) + \cdots$

$$g(t^*) = g(t_n) + \varepsilon_n g'(t_n) + \frac{{\varepsilon_n}^2}{2!} g''(t_n) + \cdots \tag{2.3}$$

By neglecting higher power ε_n, i.e. neglect terms from ${\varepsilon_n}^3$ onwards. Using (2.2) and (2.3) , we have

$$\varepsilon_n^2 g''(t_n) + 2\varepsilon_n g'(t_n) + 2g(t_n) = 0\,,$$

$$\varepsilon_n = \left[-2g'(t_n) \pm \sqrt{4g'(t_n) - 8g(t_n)\, g''(t_n)}\right] \div 2g''(t_n)\ . \tag{2.4}$$

On Substituting t^* by t_{n+1} in (2.1) and from (2.4), we get

$$t_{n+1} = t_n - \frac{2g(t_n)}{g'(t_n)} \left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right) \tag{2.5}$$

where, $\mu_n = \dfrac{g(t_n)\, g''(t_n)}{[g'(t_n)]^2}$ and

$$g''(t_n) = \frac{2}{t_{n-1} - t_n} \left[3\frac{g(t_{n-1}) - g(t_n)}{t_{n-1} - t_n} - 2g'(t_n) - g'(t_{n-1})\right]\ .$$

Here we developed a new algorithm by taking the first two steps from [4] and (2.5) as the third step.

2.1. **Algorithm.** The iterative scheme is computed by x_{n+1} as $z_n = t_n - \dfrac{g(t_n)}{g'(t_n)}$,

$$\begin{aligned} y_n &= z_n - \frac{2g\left(t_n\right)}{g^{'}\left(t_n\right) + \sqrt{\left(2g^{'}\left(z_n\right) - g^{'}\left(t_n\right)\right) g^{'}\left(t_n\right)}} \\ x_{n+1} &= y_n - \frac{2g\left(y_n\right)}{g^{'}\left(y_n\right)} \left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right), \end{aligned}$$

where $\mu_n = \dfrac{g\left(y_n\right) g^{''}\left(y_n\right)}{\left[g^{'}\left(y_n\right)\right]^2}$ and

$$g^{''}\left(y_n\right) = \frac{2}{z_n - y_n}\left[3\frac{g\left(z_n\right) - g\left(y_n\right)}{z_n - y_n} - 2g^{'}\left(y_n\right) - g^{'}\left(z_n\right)\right]. \tag{2.6}$$

The method (2.6) is known as the ninth order convergent method (MSK), it requires two functional evaluations and three first derivatives.

3. Convergence Criteria

Theorem 3.1. *Let $t_0 \in I$ be a single zero of a sufficiently differentiable function g for an open interval I. If t_0 is in the neighborhood of t^*. Then the algorithm (2.6) has tenth order convergence.*

Proof. Let the single zero of (1.1) be t^* and $t^* = t_n + \varepsilon_n$. Thus, $g\left(t^*\right) = 0$. By Taylor's series, writing $g\left(t^*\right)$ about t_n, we obtain:

$$g\left(t_n\right) = g^{'}\left(t^*\right)\left(\varepsilon_n + c_2\varepsilon_n{}^2 + c_3\varepsilon_n{}^3 + c_4\varepsilon_n{}^4 + \cdots\right) \tag{3.1}$$

$$g^{'}\left(t_n\right) = g^{'}\left(t^*\right)\left(1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n{}^2 + 4c_4\varepsilon_n{}^3 + \cdots\right) \tag{3.2}$$

Dividing (3.1) by (3.2), we get:

$$\frac{g\left(t_n\right)}{g^{'}\left(t_n\right)} = \left(\varepsilon_n - c_2\varepsilon_n{}^2 - \left(2c_3 - 2c_2{}^2\right)\varepsilon_n{}^3 - \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n{}^4 + \cdots\right).$$

From $z_n = t_n - \dfrac{g\left(t_n\right)}{g^{'}\left(t_n\right)}$, we get $z_n = t^* + \omega_n$, where

$$\omega_n = c_2\varepsilon_n{}^2 + \left(2c_3 - 2c_2{}^2\right)\varepsilon_n{}^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n{}^4 + \cdots$$

Now

$$g\left(z_n\right)=g^{'}\left(t^*\right)\left(c_2{\varepsilon_n}^2+\left(2c_3-2{c_2}^2\right){\varepsilon_n}^3+\left(3c_4-7c_2c_3+5c_2^3\right){\varepsilon_n}^4+\cdots\right)$$

$$g^{'}\left(z_n\right)=g^{'}\left(t^*\right)\left(1+2c_2^2{\varepsilon_n}^2+2c_2\left(2c_3-2{c_2}^2\right){\varepsilon_n}^3+\left(6c_2c_4-11c_2^2c_3+8c_2^4\right){\varepsilon_n}^4+\cdots\right)$$

$$2g'\left(z_n\right)-g'\left(t_n\right)=1-2c_2\varepsilon_n+\left(4c_2^2-3c_3\right)\varepsilon_n^2+\left(8c_2c_3-8c_2^3-4c_4\right)\varepsilon_n^3+\dots$$

$$\left(g^{'}\left(t_n\right)+\sqrt{\left(2g^{'}\left(z_n\right)-g^{'}\left(t_n\right)\right)g^{'}\left(t_n\right)}\right)^{-1}$$

$$=\frac{1}{2}\left(1-c_2\varepsilon_n+\left(-\frac{3}{2}c_3+c_2^2\right){\varepsilon_n}^2+\cdots\right)\frac{2g\left(t_n\right)}{g^{'}\left(t_n\right)+\sqrt{\left(2g^{'}\left(z_n\right)-g^{'}\left(t_n\right)\right)g^{'}\left(t_n\right)}}$$

$$=\varepsilon_n+k_1{\varepsilon_n}^3+k_2{\varepsilon_n}^4+\cdots$$

where $k_1=\left(\frac{-1}{2}c_3\right),k_2=\left(\frac{5}{2}c_2c_3-c_4\right),\cdots$

From the second step in the scheme (2.6), we have $y_n=t^*+Y$, where $Y=\left(k_1{\varepsilon_n}^3+k_2{\varepsilon_n}^4+\cdots\right)$.

$$g\left(y_n\right)=g^{'}\left(t^*\right)\left(Y+c_2Y^2+c_3Y^3+c_4Y^4+\cdots\right) \tag{3.3}$$

$$g^{'}\left(y_n\right)=g^{'}\left(t^*\right)\left(1+2c_2Y+3c_3Y^2+4c_4Y^3+\cdots\right). \tag{3.4}$$

Now, we have

$$g^{''}\left(y_n\right)=g^{'}\left(t^*\right)\left(2c_2+3c_2c_3\left(2c_2^2+\frac{1}{2}c_3\right)\varepsilon_n^3+\cdots\right).$$

From $\mu_n=\dfrac{g\left(y_n\right)g^{''}\left(y_n\right)}{\left[g^{'}\left(y_n\right)\right]^2}$, we get

$$\mu_n=P_1\varepsilon_n^3+P_2\varepsilon_n^4+\cdots \tag{3.5}$$

where, $P_1=2c_2\left(2c_2^2+\frac{1}{2}c_3\right),P_2=2c_2\left(-9c_2^3+\frac{9}{2}c_2c_3+c_4\right)\cdots$

Using (3.5), we get

$$\left(1+\sqrt{1-2\mu_n}\right)^{-1}=\frac{1}{2}\left(1+\frac{P_1}{2}\varepsilon_n^3+\frac{P_2}{2}\varepsilon_n^4+\cdots\right) \tag{3.6}$$

On dividing (3.3) and (3.4),

$$\frac{g\left(y_n\right)}{g^{'}\left(y_n\right)}=\left(Y-c_2Y^2-\left(2c_3-2{c_2}^2\right)Y^3-\left(3c_4-7c_2c_3+4c_2^3\right)Y^4+\cdots\right) \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{2g\left(y_n\right)}{g^{'}\left(y_n\right)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)=y_n+\left(c_2^2-2c_3\right)\left(\frac{-1}{2}c_3\right)^3\varepsilon_n^9+o\left(\varepsilon_n^{10}\right).$$

From the third step of (2.6), i.e. $x_{n+1} = y_n - \frac{2g(y_n)}{g'(y_n)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)$, we get

$$\varepsilon_{n+1} = \left(c_2^2 - 2c_3\right)\left(\frac{-1}{8}c_3^3\right)\varepsilon_n^9 + o\left(\varepsilon_n^{10}\right) .$$

Thus, it's proved that this new scheme is ninth order convergence and its efficiency index is $\sqrt[5]{9}$=1.5518. □

4. Numerical Examples

We consider some examples considered by Vatti et.al, [7] and Mylapalli et.al, [3] and compared our method with NR, FA, SK, ZHONG methods. The computations are carried out by using mpmath-PYTHON and the number of iterations for these methods are obtained for comparisons such that $|x_{n+1} - x_n| < 10^{-59}$ and $|g(x_{n+1})| < 10^{-201}$. The test functions and simple zeros are given below:

$$\begin{aligned} h_1(x) &= \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, t^* = -0.7848959876612125 \\ h_2(x) &= \sin x + \cos x + x, t^* = -0.4566247045676308 \\ h_3(x) &= (x+2)e^x - 1, t^* = -0.442854010023885 \\ h_4(x) &= x^3 - 10, t^* = 2.1544346900318837 . \end{aligned}$$

A chemical equilibrium problem: Consider the equation from [5] describing the fraction of the nitrogen hydrogen feed that gets converted to ammonia (this fraction is called fractional conversion) in polynomial form as:

$$\begin{aligned} h_5(x) &= x^4 - 7.79075x^3 + 2.511x - 1.674, t^* = 0.2777595428417206 \\ h_6(x) &= x^3 + 4x^2 - 10, t^* = 1.365230013414096 . \end{aligned}$$

Where P is the order of convergence, N is the number of functional values per iteration and EI is the Efficiency Index. Where x_0 is the initial approximation, n is the number of iterations, er is the error and fv is the functional value.

5. Conclusion

Here, in this scheme, we introduced a new ninth order convergent iterative method with efficiency index 1.5518. It requires two functional evaluations and three first derivatives. Table 1 compares the efficiency of different methods and

TABLE 1. Analogy of efficiency

Methods	P	N	EI
NR	2	2	1.414
FA	9	6	1.442
SK	9	5	1.551
ZONG	9	5	1.551
MSK	9	5	1.551

TABLE 2. Analogy of different methods

h	Method	x_0	n	er	fx	x_0	n	er	fx
h_1	NR	-1	7	9.0(63)	2.5(63)	-0.5	8	6.4(71)	1.8(70)
	FA		8	3.2(112)	9.1(12)		7	4.1(77)	1.1(76)
	SK		3	3.7(81)	5.2(82)		DIVERGENT		
	ZONG		3	6.4(84)	3.3(80)		4	1.2(200)	4.0(201)
	MSK		3	3.6(77)	1.0(76)		3	1.7(67)	4.9(67)
h_2	NR	0.1	7	1.8(69)	4.4(69)	-1	7	1.8(69)	4.4(69)
	FA		5	4.9(99)	1.1(98)		5	3.4(122)	8.0(122)
	SK		3	1.4(76)	3.4(76)		3	1.6(87)	3.9(87)
	ZONG		3	2.5(80)	6.0(80)		3	8.8(114)	2.0(113)
	MSK		3	8.9(69)	2.0(68)		3	2.6(71)	6.1(71)
h_3	NR	-0.2	8	3.6(103)	5.9(103)	-0.8	8	4.3(73)	7.1(73)
	FA		5	5.5(112)	9.0(112)		5	2.4(98)	3.9(98)
	SK		3	2.1(71)	3.1(71)		4	1.2(201)	4.2(201)
	ZONG		4	2.5(200)	4.0(200)		4	1.9(200)	4.0(200)
	MSK		3	1.2(86)	2.0(86)		3	3.3(77)	5.5(77)
h_4	NR	1.9	8	9.5(115)	1.3(113)	3	8	3.6(66)	5.0(67)
	FA		5	1.5(84)	2.4(83)		6	1.2(90)	7.2(89)
	SK		3	1.2(78)	1.7(77)		4	1.9(200)	2.0(199)
	ZONG		3	2.2(80)	1.2(78)		4	1.2(199)	2.0(199)
	MSK		3	1.4(99)	2.0(98)		3	5.1(78)	7.1(77)
h_5	NR	0.1	8	1.2(79)	1.4(78)	0.4	7	1.3(67)	1.2(66)
	FA		6	9.0(104)	8.1(102)		5	7.4(88)	6.6(87)
	SK		DIVERGENT				3	1.8(74)	1.6(73)
	ZONG		4	5.7(201)	8.3(200)		3	1.6(70)	1.5(69)
	MSK		3	1.3(82)	1.1(82)		3	1.0(92)	9.2(92)
H_6	NR	1	8	2.8(88)	4.7(87)	1.8	8	9.4(95)	1.4(93)
	FA		6	1.5(131)	2.6(130)		5	6.5(65)	6.2(62)
	SK		3	3.4(64)	5.6(63)		3	3.3(68)	1.4(64)
	ZONG		4	1.1(199)	1.0(198)		4	9.1(200)	5.7(199)
	MSK		3	1.6(87)	2.7(86)		3	1.6(87)	2.7(86)

the computational results in Table 2 show the dominance of MSK over NR, FA, SK, ZHONG methods.

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