Advances in Mathematics: Scientific Journal **9** (2020), no.7, 5291–5298
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.7.98

A THREE-STEP NINTH ORDER ITERATIVE METHOD FOR SOLVING NON-LINEAR EQUATIONS

MANI SANDEEP KUMAR MYLAPALLI [1], RAJESH KUMAR PALLI, AND RAMADEVI SRI

ABSTRACT. The scope of this paper is to establish a new ninth order iterative method to find the root of non-linear equations. In this paper we came up with a new modification of Newton's method with higher-order convergence and here itself we proved that the order of convergence is ninth. Finally, we tested with several problems to show the efficiency of the method over the existing ones.

1. INTRODUCTION

In oresent days much development happening on solving non-linear scalar equations of the form

$$g(t) = 0\,. \tag{1.1}$$

Newton's method (NR) [2] is a one of the optimal second order method to obtain the root of non-linear scalar equation and is given by

$$t_{n+1} = t_n - \frac{g(t_n)}{g'(t_n)} n = 0, 1, 2, \cdots$$

and the NR method converges quadratically and its efficiency index is $\sqrt{2} = 1.414$.

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 65F08.
Key words and phrases. Iterative method, non-linear scalar equation, functional evolutions, convergence analysis, efficiency index.

An iterative method with ninth order convergence (ZONG) for solving non-linear equations proposed by Zhongyong et.al, [7] is given by

$$\begin{aligned} y_n &= t_n - \frac{g(t_n)}{g'(t_n)} \\ z_n &= y_n - \left\{1 + \left(\frac{g(y_n)}{g(t_n)}\right)^2\right\} \frac{g(y_n)}{g'(y_n)} \\ t_{n+1} &= z_n - \left\{1 + 2\left(\frac{g(y_n)}{g(t_n)}\right)^2 + 2\frac{g(z_n)}{g(y_n)}\right\} \frac{g(z_n)}{g'(y_n)}. \end{aligned}$$

A quadrature based three-step ninth-order iterative method (SK) proposed by Khattri [5] is given by

$$\begin{aligned} y_n &= t_n - \frac{g(t_n)}{g'(t_n)} \\ z_n &= y_n - \frac{(t_n - y_n)\, g(y_n)}{g(t_n) - 2g(y_n)} \\ x_{n+1} &= z_n - \frac{g(z_n)\, g'(z_n)}{(g'(z_n))^2 - g(z_n)\left[\frac{g(z_n) - g(t_n) - g'(t_n)(z_n - t_n)}{(z_n - t_n)^2}\right]}. \end{aligned}$$

New ninth order J-Halley method for solving non-linear equations (FA) proposed by Farooq et.al, [1] is given by

$$\begin{aligned} y_n &= t_n - \frac{2h(t_n)}{3h'(t_n)} \\ z_n &= t_n - J_f \frac{h(t_n)}{h'(t_n)} \\ x_{n+1} &= z_n - \frac{2h(z_n)\, h'(z_n)}{2h'(z_n) - h(z_n)\, L}. \end{aligned}$$

where $J_f = \dfrac{3h'(y_n) + h'(t_n)}{6h'(y_n) - 2h'(t_n)}$ and $L = \dfrac{h'(z_n) - h'(t_n)}{z_n - t_n}$.

In section 2, we defined the new three-step iterative method and in section 3, we concluded our method is converging with order nine. Finally, in section 4, we compared our new scheme with other methods of the same order of convergence discussed in section 1.

2. NINTH ORDER CONVERGENT(SM) METHOD

Consider t^* is an exact root of (1.1) where g(t) is continuous and has well defined first derivatives. Let t_n be the root of nth approximation of (1.1) and is

$$t^* = t_n + \varepsilon_n \,, \tag{2.1}$$

where ε_n is the error. Thus, we get

$$g\left(t^*\right) = 0\,. \tag{2.2}$$

Writing $g\left(t^*\right)$ by Taylor's series about t_n, we have $g\left(t^*\right) = g\left(t_n\right) + \left(t^* - t_n\right) g^{'}\left(t_n\right) + \frac{\left(t^*-t_n\right)^2}{2!} g^{''}\left(t_n\right) + \cdots$

$$g\left(t^*\right) = g\left(t_n\right) + \varepsilon_n g^{'}\left(t_n\right) + \frac{{\varepsilon_n}^2}{2!} g^{''}\left(t_n\right) + \cdots\,, \tag{2.3}$$

by neglecting higher power ε_n, i.e. neglect terms from ${\varepsilon_n}^3$ onwards. Using (2.2) and (2.3), we have

$$\varepsilon_n^2 g^{''}\left(t_n\right) + 2\varepsilon_n g^{'}\left(t_n\right) + 2g\left(t_n\right) = 0$$

$$\varepsilon_n = \left[-2g^{'}\left(t_n\right) \pm \sqrt{4g^{'}\left(t_n\right) - 8g\left(t_n\right) g^{''}\left(t_n\right)}\right] \div 2g^{''}\left(t_n\right)\,. \tag{2.4}$$

On Substituting t^* by t_{n+1} in (2.1) and from (2.4), we get

$$t_{n+1} = t_n - \frac{2g\left(t_n\right)}{g^{'}\left(t_n\right)} \left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\,, \tag{2.5}$$

where, $\mu_n = \dfrac{g\left(t_n\right) g^{''}\left(t_n\right)}{\left[g^{'}\left(t_n\right)\right]^2}$ and

$$g^{''}\left(t_n\right) = \frac{2}{t_{n-1} - t_n}\left[3\frac{g\left(t_{n-1}\right) - g\left(t_n\right)}{t_{n-1} - t_n} - 2g^{'}\left(t_n\right) - g^{'}\left(t_{n-1}\right)\right]\,.$$

Here we developed a new algorithm by taking the first two steps from [3] and (2.5) as the third step.

2.1. **Algorithm.**

The iterative scheme is computed by x_{n+1} as $z_n = t_n - \dfrac{g(t_n)}{g'(t_n)}$

$$\begin{aligned} y_n &= z_n + \left(g^{'}\left(z_n\right) - g^{'}\left(t_n\right)\right) \frac{g\left(t_n\right)}{2\left(g^{'}\left(t_n\right)\right)^2} \\ x_{n+1} &= y_n - \frac{2g(y_n)}{g'(y_n)}\left(\frac{1}{1+\sqrt{1-2\mu n}}\right) \end{aligned}$$

where $\mu_n = \dfrac{g(y_n)\, g''(y_n)}{[g'(y_n)]^2}$ and

$$g''(y_n) = \frac{2}{z_n - y_n}\left[3\frac{g(z_n) - g(y_n)}{z_n - y_n} - 2g'(y_n) - g'(z_n)\right]. \tag{2.6}$$

The method (2.6) is known as the ninth order convergent method (SM), it requires two functional evaluations and three first-order derivatives.

3. Convergence Criteria

Theorem 3.1. *Let $t_0 \in I$ be a single zero of a sufficiently differentiable function g for an open interval I. If t_0 is in the neighborhood of t^*. Then the algorithm (2.6) has tenth order convergence.*

Proof. Let the single zero of (1.1) be t^* and $t^* = t_n + \varepsilon_n$. Thus, $g(t^*) = 0$. By Taylor's series, writing $g(t^*)$ about t_n, we obtain:

$$g(t_n) = g'(t^*)\left(\varepsilon_n + c_2{\varepsilon_n}^2 + c_3{\varepsilon_n}^3 + c_4{\varepsilon_n}^4 + \cdots\right) \tag{3.1}$$

$$g'(t_n) = g'(t^*)\left(1 + 2c_2\varepsilon_n + 3c_3{\varepsilon_n}^2 + 4c_4{\varepsilon_n}^3 + \cdots\right). \tag{3.2}$$

Dividing (3.1) by (3.2), we get:

$$\frac{g(t_n)}{g'(t_n)} = \left(\varepsilon_n - c_2{\varepsilon_n}^2 - \left(2c_3 - 2{c_2}^2\right){\varepsilon_n}^3 - \left(3c_4 - 7c_2c_3 + 4c_2^3\right){\varepsilon_n}^4 + \cdots\right).$$

From $z_n = t_n - \dfrac{g(t_n)}{g'(t_n)}$, we get $z_n = t^* + \omega_n$, where $\omega_n = c_2{\varepsilon_n}^2 + \left(2c_3 - 2{c_2}^2\right){\varepsilon_n}^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right){\varepsilon_n}^4 + \cdots$. Now

$$g(z_n) = g'(t^*)\left(c_2{\varepsilon_n}^2 + \left(2c_3 - 2{c_2}^2\right){\varepsilon_n}^3 + \left(3c_4 - 7c_2c_3 + 5c_2^3\right){\varepsilon_n}^4 + \cdots\right)$$
$$g'(z_n) = g'(t^*)\left(1 + 2c_2^2{\varepsilon_n}^2 + 2c_2\left(2c_3 - 2{c_2}^2\right){\varepsilon_n}^3 + \left(6c_2c_4 - 11c_2^2c_3 + 8c_2^4\right){\varepsilon_n}^4 + \cdots\right)$$
$$\left(g'(z_n) - g'(t_n)\right)\frac{g(t_n)}{2\left(g'(t_n)\right)^2} = \left(-c_2^2{\varepsilon_n}^2 + \left(4c_2^2 - \frac{3}{2}c_3\right){\varepsilon_n}^3 + \left(-13c_2^3 + \frac{23}{2}c_2c_3 - 2c_4\right){\varepsilon_n}^4 + \cdots\right).$$

From the second step in the scheme (2.6), we get $y_n = t^* + Y$, where

$$Y = \left(\left(2c_2^2 + \frac{1}{2}c_3\right){\varepsilon_n}^3 + \left(-9c_2^3 + \frac{9}{2}c_2c_3 + c_4\right){\varepsilon_n}^4 + \cdots\right).$$

$$g(y_n) = g'(t^*)\left(Y + c_2Y^2 + c_3Y^3 + c_4Y^4 + \cdots\right) \tag{3.3}$$

$$g'(y_n) = g'(t^*)\left(1 + 2c_2Y + 3c_3Y^2 + 4c_4Y^3 + \cdots\right) . \tag{3.4}$$

Now, we obtain

$$g''(y_n) = g'(t^*)\left(2c_2 + 3c_2c_3\left(2c_2^2 + \frac{1}{2}c_3\right)\varepsilon_n^3 + \cdots\right) .$$

From $\mu_n = \dfrac{g(y_n)\, g''(y_n)}{\left[g'(y_n)\right]^2}$, we get

$$\mu_n = P_1\varepsilon_n^3 + P_2\varepsilon_n^4 + \cdots , \tag{3.5}$$

where, $P_1 = 2c_2\left(2c_2^2 + \dfrac{1}{2}c_3\right), P_2 = 2c_2\left(-9c_2^3 + \dfrac{9}{2}c_2c_3 + c_4\right)\cdots$.
Using (3.5), we get

$$\left(1 + \sqrt{1 - 2\mu_n}\right)^{-1} = \frac{1}{2}\left(1 + \frac{P_1}{2}\varepsilon_n^3 + \frac{P_2}{2}\varepsilon_n^4 + \cdots\right) . \tag{3.6}$$

On dividing (3.3) and (3.4),

$$\frac{g(y_n)}{g'(y_n)} = \left(Y - c_2Y^2 - \left(2c_3 - 2{c_2}^2\right)Y^3 - \left(3c_4 - 7c_2c_3 + 4c_2^3\right)Y^4 + \cdots\right) \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{2g(y_n)}{g'(y_n)}\left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right) = y_n + \left(c_2^2 - 2c_3\right)\left(2c_2^2 + \frac{1}{2}c_3\right)^3 \varepsilon_n^9 + o\left(\varepsilon_n^{10}\right) ,$$

and from the third step of (2.6) ,i.e. $x_{n+1} = y_n - \frac{2g(y_n)}{g'(y_n)}\left(\frac{1}{1+\sqrt{1-2\mu n}}\right)$, we get

$$\varepsilon_{n+1} = \left(c_2^2 - 2c_3\right)\left(2c_2^2 + \frac{1}{2}c_3\right)^3 \varepsilon_n^9 + o\left(\varepsilon_n^{10}\right) .$$

Thus, it's proved that this new scheme is ninth order convergence and its efficiency index is $\sqrt[5]{9} = 1.5518$. □

4. NUMERICAL EXAMPLES

We consider the some examples considered by Vatti [6] and MMS [4] and compared our method with NR, SK, ZONG, FA methods. The computations are carried out by using mpmath-PYTHON and the number of iterations for these methods are obtained for comparisons such that $|x_{n+1} - x_n| < 10^{-59}$ and $|g(x_{n+1})| < 10^{-201}$. The test functions and simple zeros are given below

$$\begin{aligned}
h_1(x) &= \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, t^* = -0.7848959876612125 \\
h_2(x) &= \sin x + \cos x + x, t^* = -0.4566247045676308 \\
h_3(x) &= (x+2)e^x - 1, t^* = -0.442854010023885 \\
h_4(x) &= x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, t^* = -0.060960589605896 \\
h_5(x) &= \cos x - x, t^* = 0.7390851332151606 \\
h_6(x) &= x^3 - 10, t^* = 2.1544346900318837 \\
h_7(x) &= e^{-x} + \cos x, t^* = 1.7461395304080124 \\
h_8(x) &= e^{\sin x} - x + 1, t^* = 2.6306641479279036 \\
h_9(x) &= \sin^2 x - x^2 + 1, t^* = 1.404491648215341
\end{aligned}$$

Where P is the order of convergence, N is the number of functional values per

TABLE 1. Analogy Of Efficiency

Methods	P	N	EI
NR	2	2	1.414
FA	9	6	1.442
SK	9	5	1.551
ZONG	9	5	1.551
SM	9	5	1.551

iteration and EI is the Efficiency Index.

Where x_0 is the initial approximation, n is the number of iterations, er is the error and fv is the functional value.

TABLE 2. Analogy Of Different Methods

h	Method	x_0	n	er	fv	x_0	n	er	fv
h_1	NR	-1	7	9.0(63)	2.5(63)	-1.9	42	4.5(109)	1.8(109)
	FA		8	3.2(112)	9.1(12)		12	4.7(77)	1.3(76)
	SK		3	3.7(81)	5.2(82)		DIVERGENT		
	ZONG		3	6.4(84)	3.3(80)		DIVERGENT		
	SM		3	3.9(81)	1.1(80)		4	9.8(92)	2.7(91)
h_2	NR	0.1	7	1.8(69)	4.4(69)	-1	7	1.8(69)	4.4(69)
	FA		5	4.9(99)	1.1(98)		5	3.4(122)	8.0(122)
	SK		3	1.4(76)	3.4(76)		3	1.6(87)	3.9(87)
	ZONG		3	2.5(80)	6.0(80)		3	8.8(114)	2.0(113)
	SM		3	1.4(77)	3.2(77)		3	4.0(72)	9.5(72)
h_3	NR	-0.3	7	7.7(66)	3.7(77)	-1	9	5.5(92)	9.0(92)
	FA		5	2.9(126)	4.9(126)		5	1.2(84)	2.0(84)
	SK		3	1.4(89)	1.8(89)		4	1.2(201)	4.1(201)
	ZONG		3	2.2(76)	3.6(76)		4	1.8(125)	3.0(125)
	SM		3	6.3(76)	1.0(75)		4	3.6(77)	5.9(77)
h_4	NR	-0.5	8	2.6(118)	2.7(118)	-0.8	8	2.6(118)	2.7(118)
	FA		5	7.7(94)	7.9(94)		5	2.9(101)	2.9(101)
	SK		3	3.8(99)	3.9(99)		3	1.4(96)	1.5(96)
	ZONG		3	1.1(78)	1.1(78)		4	3.3(200)	4.4(201)
	SM		3	2.3(92)	2.4(92)		3	1.1(87)	1.2(87)
h_5	NR	1.4	7	1.2(67)	2.1(68)	0.5	7	1.5(78)	2.6(78)
	FA		5	1.9(93)	3.2(93)		5	1.4(111)	2.4(111)
	SK		3	1.8(79)	3.9(89)		3	1.9(89)	3.9(89)
	ZONG		3	8.2(68)	1.2(67)		3	3.2(87)	5.3(87)
	SM		3	3.2(85)	5.3(85)		3	4.3(81)	7.2(91)
h_6	NR	2	7	2.5(72)	3.6(71)	2.4	7	7.6(65)	1.0(63)
	FA		5	3.3(99)	4.7(98)		5	2.7(89)	3.8(88)
	SK		3	4.5(97)	6.3(96)		3	3.9(84)	5.4(83)
	ZONG		3	3.0(86)	4.2(85)		3	3.3(71)	4.6(70)
	SM		3	1.0(84)	1.4(83)		3	9.4(76)	1.3(74)
h_7	NR	1.4	7	2.2(93)	2.6(93)	2.1	7	1.2(73)	1.4(73)
	FA		5	1.7(113)	1.9(113)		5	2.5(101)	2.9(103)
	SK		3	9.5(86)	1.1(85)		3	5.3(71)	6.1(71)
	ZONG		3	1.2(96)	1.4(96)		3	1.1(74)	1.3(74)
	SM		3	2.9(91)	3.4(91)		3	5.1(87)	5.9(87)
h_8	NR	2.4	6	5.8(90)	1.4(89)	3.1	7	2.0(76)	5.0(76)
	FA		5	2.0(122)	4.8(122)		5	1.4(99)	3.6(99)
	SK		3	3.3(132)	8.0(132)		3	5.5(66)	1.3(82)
	ZONG		3	3.2(187)	7.7(187)		3	1.8(67)	4.5(67)
	SM		3	3.4(90)	8.3(90)		3	1.0(82)	2.5(852)
h_9	NR	1.6	8	3.1(112)	7.8(112)	1.3	8	3.0(68)	7.5(68)
	FA		8	1.6(72)	4.0(72)		8	1.0(112)	2.6(112)
	SK		3	5.4(79)	1.3(78)		3	7.1(93)	1.2(92)
	ZONG		3	4.1(78)	1.0(78)		3	6.1(81)	1.5(80)
	SM		3	3.5(72)	8.8(72)		3	9.0(84)	2.2(83)

5. CONCLUSION

Here In this scheme, we introduced a new ninth order convergent iterative method with efficiency index 1.5518. It requires two functional evaluations and three first derivatives. Table 1 compares the efficiency of different methods and the computational results in Table 2 show good results when compared with the other methods.

REFERENCES

[1] F. AHMAD, S. HUSSAIN: *New ninth order J-Halley method for solving non-linear equations*, App. Math., **2013**(4) (2013), 1709–1713.

[2] J. F. TRAUB: *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1977..

[3] M. S. MYLAPALLI, R. K. PALLI, R. SRI: *Three- step iterative method with fifth-order convergent for solving non-linear equations*, Int. Jr. of Psychosocial Rehabilitation, **24**(5) (2020), 319–324.

[4] M. S. MYLAPALLI, R. K. PALLI, R. SRI: *An Optimal Three-Step Method for solving non-linear equations*, Journal of Critical Reviews, **7**(6) (2020), 100–103.

[5] S. K. KHATTRI: *Another note on some quadrature based three-step Iterative Methods*, Numerical Algebra, **3**(3) (2013), 549–555.

[6] V. B. K. VATTI, M. S. K. MYLAPALLI: *Eighteenth Order Convergent Method for solving Non-Linear Equations*, Orient. J. Comp. Sci. and techno., **10**(1) (2017), 144–150.

[7] Z. HU, L. GUOCAI, L. TIAN: *An Iterative method with ninth order convergence for solving non-linear equations*, Int. J. Contemp. Math. Science, **6**(1) (2011), 17–23.

DEPARTMENT OF MATHEMATICS GITAM (DEEMED TO BE UNIVERSITY)
VISAKHAPATNAM, INDIA
E-mail address: mmylapal@gitam.edu

DEPARTMENT OF MATHEMATICS GITAM (DEEMED TO BE UNIVERSITY)
VISAKHAPATNAM, INDIA
E-mail address: rajeshkumar.viit@gmail.com

DEPARTMENT OF MATHEMATICS
DR. L. BULLAYYA COLLEGE
VISAKHAPATNAM, INDIA
E-mail address: ramadevisri9090@gmail.com