ALMOST NANO REGULAR SPACES

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ABSTRACT. In main aim of this paper, we introduce a new separation axiom is called almost nano regular space. Also various characterizations of almost nano regular space are given.

1. INTRODUCTION

In 2013, M. Lellis Thivagar et al., [2] introduced a nano topological space with respect to a subset $X$ of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are not suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space. In main aim of this paper, we introduce a new separation axiom is called almost nano regular space. Also various characterizations of almost nano regular space are given.

2. PRELIMINARIES

Definition 2.1. [3] Let $U$ be a non-empty finite set of objects called the universe and $R$ be an equivalence relation on $U$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U, R)$ is said to be the approximation space. Let $X \subseteq U$.

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The lower approximation of \( X \) with respect to \( R \) is the set of all objects, which can be for certain classified as \( X \) with respect to \( R \) and it is denoted by \( L_R(X) \). That is, \( L_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \subseteq X \} \), where \( R(x) \) denotes the equivalence class determined by \( x \).

The upper approximation of \( X \) with respect to \( R \) is the set of all objects, which can be possibly classified as \( X \) with respect to \( R \) and it is denoted by \( U_R(X) \). That is, \( U_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \cap X \neq \emptyset \} \).

The boundary region of \( X \) with respect to \( R \) is the set of all objects, which can be classified neither as \( X \) nor as \( \neg X \) with respect to \( R \) and it is denoted by \( B_R(X) \). That is, \( B_R(X) = U_R(X) - L_R(X) \).

**Definition 2.2.** [2] Let \( U \) be the universe, \( R \) be an equivalence relation on \( U \) and \( \tau_R(X) = \{ U, \emptyset, L_R(X), U_R(X), B_R(X) \} \) where \( X \subseteq U \). Then \( \tau_R(X) \) satisfies the following axioms;

(i) \( U \) and \( \emptyset \) are in \( \tau_R(X) \),

(ii) The union of the elements of any subcollection of \( \tau_R(X) \) is in \( \tau_R(X) \),

(iii) The intersection of the elements of any finite subcollection of \( \tau_R(X) \) is in \( \tau_R(X) \).

Thus \( \tau_R(X) \) is a topology on \( U \) called the nano topology with respect to \( X \) and \((U, \tau_R(X))\) is called the nano topological space. The elements of \( \tau_R(X) \) are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by \((U, \mathcal{N})\), where \( \mathcal{N} = \tau_R(X) \). The nano-interior and nano-closure of a subset \( A \) of \( U \) are denoted by \( nint(A) \) and \( ncl(A) \) respectively.

**Definition 2.3.** [2] A subset \( H \) of a space \((U, \mathcal{N})\) is called nano \( \alpha \)-open (briefly n\( \alpha \)-open) if \( H \subseteq nint(ncl(nint(H))) \).

The complements of the above mentioned set is called their respective closed set.

**Definition 2.4.** [2] A subset \( H \) of a space \((U, \mathcal{N})\) is called nano regular closed (briefly nr-closed) if \( H = ncl(nint(H)) \).

The complements of the above mentioned set are called their respective an open set.
**Definition 2.5.** [1] A subset $H$ of a nano topological space $(U, \mathcal{N})$ is called a nano t-set if $\text{nint}(H) = \text{nint}(\text{ncl}(H))$.

**Definition 2.6.** [4] A subset $H$ of a nano topological space $(U, \mathcal{N})$ is said to be nano semi regular if $H$ is nano semi open and a nano t-set.

### 3. Almost Nano Regular Spaces

**Definition 3.1.** A space $(U, \mathcal{N})$ is said to be almost nano regular (briefly almost $n$-regular), if for each $n$-regular closed set $H$ and a point $a \in U - H$, there exist disjoint $n$-open sets $S_1$ and $S_2$ such that $a \in S_1$ and $H \subseteq S_2$.

**Theorem 3.1.** For a nano topological space $(U, \mathcal{N})$, the following are equivalent:

(i) $(U, \mathcal{N})$ is almost $n$-regular.

(ii) For each point $x \in U$ and each $n$-regular open set $F$ containing $x$, there exists a $n$-regular open set $E$ such that $x \in E \subseteq \text{ncl}(E) \subseteq F$.

(iii) For each point $x \in U$ and each neighbourhood $J$ of $x$, there exists a $n$-regular open neighbourhood $F$ of $x$ such that $\text{ncl}(F) \subseteq \text{nint}(\text{ncl}(J))$.

(iv) For each point $x \in U$ and each neighbourhood $J$ of $x$ there exists a $n$-open neighbourhood $F$ of $x$ such that $\text{ncl}(F) \subseteq \text{nint}(\text{ncl}(J))$.

(v) For every $n$-regular closed set $A$ and each point $x$ not belonging to $A$, there exist $n$-open sets $E$ and $F$ such that $x \in E$, $A \subseteq F$ and $\text{ncl}(E) \cap \text{ncl}(F) = \emptyset$.

(vi) Every $n$-regular closed set $L$ is expressible as an intersection of some $n$-regular closed neighbourhood of $L$.

(vii) Every $n$-regular closed set $L$ is identical with the intersection of all $n$-closed neighbourhoods of $L$.

(viii) For every set $G$ and every $n$-regular open set $H$ such that $G \cap H \neq \emptyset$, there exists a $n$-open set $M$ such that $G \cap M \neq \emptyset$ and $\text{ncl}(M) \subseteq H$.

(ix) For every non-empty set $G$ and every $n$-regular closed set $H$ satisfying $G \cap H \neq \emptyset$, there exist disjoint $n$-open sets $M$ and $N$ such that $G \cap M \neq \emptyset$ and $H \subseteq N$.

**Proof.** $(i) \Rightarrow (ii)$. If $F$ is a $n$-regular open set containing $x$, then $\{x\}$ is disjoint from the $n$-regular closed set $U - V$. Therefore, there exist $n$-open sets $E_1$ and $E_2$ such that $\{x\} \subseteq E_1$, $U - F \subseteq E_2$ and $E_1 \cap E_2 = \emptyset$. Then $\text{ncl}(E_1) \cap E_2$ is also empty and hence $\text{ncl}(E_1) \subseteq U - E_2 \subseteq F$. Thus, $x \in E_1 \subseteq \text{ncl}(E_1) \subseteq F$. 


Again, \( E_1 \subseteq \text{nint}(\text{ncl}(E_1)) \subseteq \text{ncl}(E_1) \subseteq F \). Therefore, if \( \text{nint}(\text{ncl}(E_1)) = E \), then \( E_1 \subseteq E \subseteq \text{ncl}(E) = \text{ncl}(E_1) \subseteq F \). Hence \( x \in E \subseteq \text{ncl}(E) \subseteq F \) where \( E \) is \( n \)-regular open.

(ii) \( \Rightarrow \) (iii). Obvious.

(iii) \( \Rightarrow \) (iv). Obvious.

(iv) \( \Rightarrow \) (v). If \( G \) is \( n \)-regular closed and \( x \notin G \), then \( U - G \) is a neighbourhood of \( x \). Therefore, there exists a \( n \)-open set \( F \) such that \( x \in F \subseteq \text{ncl}(F) \subseteq U - G \). Again, \( F \) is a neighbourhood of \( x \). Therefore there exists a \( n \)-open set \( E \) such that \( x \in E \subseteq \text{ncl}(E) \subseteq F \). Then \( E \) and \( U - \text{ncl}(F) \) are \( n \)-open sets with disjoint nano closures containing \( x \) and \( G \) respectively.

(v) \( \Rightarrow \) (vi). If \( L \) is a \( n \)-regular open set, then for each \( x \notin L \), there exist \( n \)-open sets \( M_x \) and \( N_x \) such that \( L \subseteq M_x \), \( x \in N_x \) and \( \text{ncl}(N_x) \cap \text{ncl}(M_x) = \phi \). Thus, \( L \subseteq M_x \) and \( x \notin \text{ncl}(M_x) \). It can be seen easily that \( L = \bigcap \text{ncl}(M_x) \). Also, each \( \text{ncl}(M_x) \) is a \( n \)-regular closed neighbourhood of \( L \).

(vi) \( \Rightarrow \) (vii). Obvious.

(vii) \( \Rightarrow \) (viii). Let \( G \) be any set and let \( H \) be \( n \)-regular open such that \( G \cap H \neq \phi \). Then, there exists a point \( x \in G \cap H \). Therefore, \( U - H \) is a \( n \)-regular closed set and hence \( U - H = \bigcap_{i \in I} J_i \) where \( \{ J_i : i \in I \} \) is the family of \( n \)-closed neighbourhoods of \( U - H \). Since \( x \in H \), therefore \( x \notin \bigcap J_i \) and thus \( x \notin J_i \) for all \( i \). Since \( J_i \) is a neighbourhood of \( U - H \), therefore there exists a \( n \)-open set \( N \) such that \( U - H \subseteq N \subseteq J_i \). Let \( V = U - J_i \). Then \( V \) is a \( n \)-open set containing \( x \) and also \( x \in G \). Thus \( x \in V \cap G \), that is, \( V \cap G \neq \phi \). Also \( U - N \) is \( n \)-closed, therefore \( V = \text{ncl}(U - J_i) \subseteq U - N \subseteq H \).

(viii) \( \Rightarrow \) (ix). Obvious.

(ix) \( \Rightarrow \) (1). If \( G \) is \( n \)-regular closed and \( x \notin G \), then \( \{ x \} \) is non-empty and \( G \cap \{ x \} = \phi \). Therefore there exist disjoint \( n \)-open sets \( M \) and \( N \) such that \( \{ x \} \cap M = \phi \) and \( G \subseteq N \). Thus \( x \in M, G \subseteq N \) and \( M \cap N = \phi \). Hence \( U \) is almost \( n \)-regular.

Theorem 3.2. Every almost \( n \)-regular, \( n \)-semi regular space is \( n \)-regular.

Proof. Suppose \( U \) is an almost \( n \)-regular, \( n \)-semi regular space. Let \( J \) be any \( n \)-open set containing a point \( x \in U \). Since \( U \) is \( n \)-semi regular, there exists a \( n \)-open set \( E \) such that \( x \in E \subseteq \text{nint}(\text{ncl}(E)) \subseteq J \). Again, since \( E \) is a \( n \)-open neighbourhood of \( x \) and \( U \) is almost \( n \)-regular, there is a \( n \)-open set \( F \) such that
Proof. Suppose that \( x \in U \subseteq \text{ncl}(F) \subseteq \text{nint}(\text{ncl}(E)). \) Hence \( x \in F \subseteq \text{ncl}(F) \subseteq E. \) Thus \( U \) is \( n \)-regular. \( \Box \)

**Theorem 3.3.** Let \((U, \mathcal{N})\) is almost \( n \)-regular if and only if for each \( x \in U \) and each \( n \)-regular open neighbourhood \( E \) of \( x \), there exists a \( n \)-regular open set \( F \) such that \( x \in F \subseteq \text{ncl}(F) \subseteq E. \)

Proof. Suppose that \( U \) is an almost \( n \)-regular space. Let \( x \in U \) and let \( F \) be a \( n \)-regular open set containing \( x \). But then \( U - F \) is a \( n \)-regular closed set with \( x \notin U - F \). But \( U \) is almost \( n \)-regular. Hence there exists disjoint \( n \)-regular open sets \( F \) and \( T \) such that \( x \in F \) and \( U - E \subseteq T \). But then \( F \subseteq U - T \) and \( U - T \subseteq E \). Therefore, \( x \in F \subseteq \text{ncl}(F) \subseteq E. \)

Conversely, suppose that for each \( x \in U \) and for each \( n \)-regular open neighbourhood \( E \) of \( x \), there is a \( n \)-regular open set \( F \) such that \( x \in F \subseteq \text{ncl}(F) \subseteq E. \) Let \( L \) be a \( n \)-regular closed set in \( U \) and let \( x \notin L \). Then there exists a \( n \)-regular open set \( F \) such that \( x \in F \subseteq \text{ncl}(F) \subseteq U - L \) implies \( x \in F \) and \( L \subseteq U - \text{ncl}(F) \). Then the sets \( F \) and \( U - \text{ncl}(F) \) are \( n \)-regular open sets with \( F \subseteq \text{ncl}(F) \). Therefore \( F \) and \( U - F \) are disjoint. Hence \( U \) is almost \( n \)-regular. \( \Box \)

**Theorem 3.4.** Let \( P \) and \( Q \) be non-empty nano topological spaces. The product \( P \times Q \) is almost \( n \)-regular if and only if both \( P \) and \( Q \) are almost \( n \)-regular.

Proof. Suppose that \( P \times Q \) is almost \( n \)-regular. Let \( p_0 \in P \). Let \( E \) be any \( n \)-open neighbourhood of \( p_0 \). Let \( q_0 \in Q \). Then \( E \times Q \) is a \( n \)-regular open neighbourhood of \((p_0, q_0)\). But \( P \times Q \) is almost \( n \)-regular. Hence there exists a \( n \)-regular open set \( T \) in \( P \times Q \) such that \((p_0, q_0) \in T \subseteq \text{ncl}(T) \subseteq E \times Q \). Let \( H_1 \times H_2 \) be a basic \( n \)-regular open subset of \( P \times Q \) such that \((p_0, q_0) \in H_1 \times H_2 \subseteq T \). Then \((p_0, q_0) \in H_1 \times H_2 \subseteq \text{ncl}(H_1 \times H_2) \subseteq \text{ncl}(T) \subseteq E \times Q \). Implies that \((p_0, q_0) \in H_1 \times H_2 \subseteq \text{ncl}(H_1) \times \text{ncl}(H_2) \subseteq E \times Q \). Therefore \( p_0 \in H_1 \subseteq \text{ncl}(H_1) \subseteq E \). Hence \( P \) is almost \( n \)-regular. Similarly we can show that \( Q \) is almost \( n \)-regular.

Conversely, suppose that \( P \) and \( Q \) are almost \( n \)-regular. Let \((p_0, q_0) \in P \times Q \). Let \( T \) be a \( n \)-regular open neighbourhood of \((p_0, q_0)\). Then there exists a basic \( n \)-regular open set \( E_1 \) and \( H_1 \) such that \( p_0 \in E_1 \subseteq \text{ncl}(E_1) \subseteq E \) and \( q_0 \in H_1 \subseteq \text{ncl}(H_1) \subseteq H \). Hence \((p_0, q_0) \in E_1 \times H_1 \subseteq \text{ncl}(E_1) \times \text{ncl}(H_1) \subseteq E \times H \subseteq T \). Implies that \((p_0, q_0) \in E_1 \times H_2 \subseteq \text{ncl}(E_1 \times H_2) \subseteq E \times H \subseteq T \). Hence \( P \times Q \) is almost \( n \)-regular. \( \Box \)
Theorem 3.5. If $J$ and $K$ are subsets of $(U, N)$, where $J$ and $K$ are $n$-regular open and $J \subseteq K$ then $\text{nc}(J) \subseteq \text{nc}(\text{nint}(\text{nc}(K)))$.

Proof. Let $J \subseteq K$. Since $J \subseteq \text{nc}(K)$ then $J \subseteq \text{nint}(\text{nc}(K))$ because $J$ is $n$-regular open. Hence $\text{nc}(J) \subseteq \text{nc}(\text{nint}(\text{nc}(K)))$. \hfill \Box

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