NANO $I_g$-NORMAL AND NANO $I_g$-REGULAR SPACES

R. PREMUKMAR$^1$, M. RAMESHPANDI, AND S. ANTONY DAVID

Abstract. In this paper a new classes of $nI_g$-normal space and $nI_g$-regular space are introduced and various characterizations and properties are given. Further, we define a new notion is called $nI_{rg}$-closed set and establish their various characteristic properties are given.

1. Introduction

In 2013, Lellis Thivagar et al., [3] introduced a nano topological space. Then the notions of an ideal nano topological space was introduced by Parimala et al., [4, 6]. A nano topological space $(U, N)$ with an ideal $I$ on $U$ is called [6] an ideal nano topological space and is denoted by $(U, N, I)$. For background material, papers [1] to [10] may be perused. A subset $H$ of a space $(U, N)$ is called $(n\alpha$-open, $nr$-open and $np$-open [3]), $nI$-open [4], $ng$-closed [2], $n\alpha g$-closed [10], $nr g$-closed [9] and $nI_g$-closed [5]. The family of all $n\alpha$-closed (resp. $N^\alpha$) and the family of all $n\star$-closed (resp. $N^\star$).

In this paper a new classes of $nI_g$-normal space and $nI_g$-regular space are introduced and various characterizations and properties are given. Further, we define a new notion is called $nI_{rg}$-closed set and establish their various characteristic properties are given.

$^1$Corresponding author
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2. Preliminaries

In the rest of the paper, we denote a nano topological space by \((U, \mathcal{N})\), where \(\mathcal{N} = \tau_R(X)\). The nano-interior and nano-closure of a subset \(A\) of \(U\) are denoted by \(\text{nint}(A)\) and \(\text{ncl}(A)\) respectively. A nano topological space \((U, \mathcal{N})\) with an ideal \(I\) on \(U\) is called \([6]\) an ideal nano topological space and is denoted by \((U, \mathcal{N}, I)\) in future is referred as a space.

Definition 2.1. A subset \(H\) of a space \((U, \mathcal{N}, I)\) is called;

(i) nano \(\ast\)-closed (briefly \(n\ast\)-closed) \([5]\) if \(H^\ast \subseteq H\),
(ii) nano dense (briefly \(n\)-dense) \([7]\) if \(\text{ncl}(H) = U\).

Definition 2.2. \([1]\) A space \((U, \mathcal{N})\) is said to be almost nano regular (briefly, almost \(n\)-regular) if for each \(n\)-regular closed set \(H\) and a point \(a \in U - H\), there exist disjoint \(n\)-open sets \(M\) and \(N\) such that \(a \in M\) and \(H \subseteq N\).

Definition 2.3. \([8]\) A space \(U\) is called nano-\(T_1\) space (briefly \(nT_1\)-space) for \(x, y \in U\) and \(x \neq y\), there exists a nano-open sets \(G\) and \(H\) such that \(x \in G\), \(y \notin G\), and \(y \in H\), \(x \notin H\).

3. \(nI_g\)-NORMAL AND \(nI_g\)-REGULAR SPACES

Definition 3.1. An ideal nanotopological space \((U, \mathcal{N}, I)\) is said to be a nano \(I_g\)-normal space(briefly \(nI_g\)-normal space) if for every pair of disjoint \(n\)-closed sets \(A\) and \(B\), there exist disjoint \(nI_g\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

Definition 3.2. An ideal nano \(I\) is said to be nano completely codense if \(\text{PO}(U) \cap I = \{\phi\}\), where \(\text{PO}(U)\) is the family of all \(np\)-open set in \((U, \mathcal{N})\).

Theorem 3.1. Let \((U, \mathcal{N}, I)\) be an ideal nanotopological space. Then the following are equivalent.

(i) \(U\) is \(nI_g\)-normal.
(ii) For every pair of disjoint \(n\)-closed sets \(E\) and \(F\), there exist disjoint \(nI_g\)-open sets \(G\) and \(H\) such that \(E \subseteq F\) and \(F \subseteq H\).
(iii) For every \(n\)-closed set \(E\) and a \(n\)-open set \(H\) containing \(E\), there exists a \(nI_g\)-open set \(G\) such that \(E \subseteq G \subseteq \text{ncl}^\ast(G) \subseteq H\).
Proof. (i) ⇒ (ii) The proof follows from the definition of $nI_g$-normal spaces.

(ii) ⇒ (iii) Let $E$ be a $n$-closed set and $H$ be a $n$-open set containing $E$. Since $E$ and $U - H$ are disjoint $n$-closed sets, there exist disjoint $nI_g$-open sets $G$ and $S$ such that $E \subseteq G$ and $U - H \subseteq S$. Again, $G \cap S = \phi$ implies that $G \cap \text{nint}^*(S) = \phi$ and so $\text{ncl}^*(G) \subseteq U - \text{nint}^*(S)$. Since $U - H$ is $n$-closed and $S \subseteq H$. Thus, we have $E \subseteq G \subseteq \text{ncl}^*(G) \subseteq U - \text{nint}^*(S) \subseteq H$ which proves (iii).

(iii) ⇒ (i) Let $E$ and $F$ be two disjoint $n$-closed subsets of $U$. By hypothesis, there exists a $nI_g$-open set $G$ such that $E \subseteq G \subseteq \text{ncl}^*(G) \subseteq U - F$. If $S = U - \text{ncl}^*(G)$, then $G$ and $S$ are the required disjoint $nI_g$-open sets containing $E$ and $F$ respectively. So, $(U, \mathcal{N}, I)$ is $nI_g$-normal.

**Theorem 3.2.** Let $(U, \mathcal{N}, I)$ be an ideal nanotopological space, where $I$ is nano completely codense. If $(U, \mathcal{N}, I)$ is $nI_g$-normal then normal space.

Proof. Suppose that $I$ is nano completely codense. By Theorem 3.1, $(U, \mathcal{N}, I)$ is $nI_g$-normal if and only if for each pair of disjoint $n$-closed sets $E$ and $F$, there exist disjoint $nI_g$-open sets $G$ and $H$ such that $E \subseteq G$ and $F \subseteq H$ if and only if $G$ is normal.

**Theorem 3.3.** Let $(U, \mathcal{N}, I)$ be a $nI_g$-normal space. If $P$ is $n$-closed and $E$ is a $ng$-closed set such that $E \cap P = \phi$, then there exist disjoint $nI_g$-open sets $G$ and $H$ such that $E \subseteq G$ and $P \subseteq H$.

Proof. Since $E \cap P = \phi$, $E \subseteq U - P$ where $U - P$ is $n$-open. Therefore, by hypothesis, $\text{ncl}(E) \subseteq U - P$. Since $\text{ncl}(E) \cap P = \phi$ and $U$ is $nI_g$-normal, there exist disjoint $nI_g$-open sets $G$ and $H$ such that $\text{ncl}(E) \subseteq G$ and $P \subseteq H$.

**Theorem 3.4.** Let $(U, \mathcal{N}, I)$ be a normal ideal nanotopological space which is $nI_g$-normal. Then the following hold.

(i) For every $n$-closed set $E$ and every $ng$-open set $F$ containing $E$, there exists a $nI_g$-open set $G$ such that $E \subseteq \text{nint}^*(G) \subseteq G \subseteq F$.

(ii) For every $ng$-closed set $E$ and every $n$-open set $F$ containing $E$, there exists a $nI_g$-closed set $G$ such that $E \subseteq G \subseteq \text{ncl}^*(G) \subseteq F$.

Proof. (i) Let $E$ be a $n$-closed set and $F$ be a $ng$-open set containing $E$. Then $E \cap (U - F) = \phi$, where $E$ is $n$-closed and $U - F$ is $ng$-closed. By Theorem 3.3, there exist disjoint $nI_g$-open sets $G$ and $H$ such that $E \subseteq G$ and $U - F \subseteq H$. 


Since $G \cap H = \phi$, we have $G \subseteq U - H$, then $E \subseteq \text{nint}^*(G)$. Therefore, $E \subseteq \text{nint}^*(G) \subseteq G \subseteq U - H \subseteq F$. This proves (i).

(ii) Let $E$ be a ng-closed set and $F$ be a n-open set containing $E$. Then $U - F$ is a n-closed set contained in the ng-open set $U - E$. By (i), there exists a $nI_g$-open set $H$ such that $U - F \subseteq \text{nint}^*(H) \subseteq H \subseteq U - E$. Therefore, $E \subseteq U - H \subseteq \text{ncl}^*(U - H) \subseteq F$. If $G = U - H$, then $E \subseteq G \subseteq \text{ncl}^*(G) \subseteq F$ and so $G$ is the required $nI_g$-closed set.

**Definition 3.3.** An ideal nanotopological space $(U, N, I)$ is said to be nano $gI$-normal (briefly $ngI$-normal) if for each pair of disjoint $nI_g$-closed sets $E$ and $F$, there exist disjoint $n$-open sets $G$ and $H$ in $U$ such that $E \subseteq G$ and $F \subseteq H$.

**Theorem 3.5.** In an ideal nanotopological space $(U, N, I)$, every n-closed set is $nI_g$-closed.

**Proof.** Obvious. \hfill \Box

**Proposition 3.1.** In an ideal nanotopological space $(U, N, I)$, every $ngI$-normal space is normal. But the converse is not true as seen from the following Example.

**Example 1.** Let $U = \{n_1, n_2, n_3, n_4\}$ with $U/R = \{\phi, \{n_1\}, U\}$, $X = \{n_1\}$ and $I = \{\phi, \{n_1\}\}$. Since $\{n_1\}_n^* = \phi$, every $n$-open set is $n^*$-closed and so every subset of $U$ is $nI_g$-closed. Now $A = \{n_1, n_2\}$ and $B = \{n_3, n_4\}$ are disjoint $nI_g$-closed sets, but they are not separated by disjoint $n$-open sets. So $(U, N, I)$ is not $ngI$-normal. Since there is no pair of disjoint $n$-closed sets, $(U, N, I)$ is normal.

**Theorem 3.6.** In an ideal nanotopological space $(U, N, I)$, the following are equivalent.

(i) $U$ is $ngI$-normal.

(ii) For every $nI_g$-closed set $E$ and every $nI_g$-open set $F$ containing $E$, there exists a $n$-open set $G$ of $U$ such that $E \subseteq G \subseteq \text{ncl}(G) \subseteq F$.

**Proof.**

(i) $\Rightarrow$ (ii). Let $E$ be a $nI_g$-closed set and $F$ be a $nI_g$-open set containing $E$. Since $E$ and $U - F$ are disjoint $nI_g$-closed sets, there exist disjoint $n$-open sets $G$ and $H$ such that $E \subseteq G$ and $U - F \subseteq H$. Now $G \cap H = \phi \Rightarrow \text{ncl}(G) \subseteq U - H$. Therefore, $E \subseteq G \subseteq \text{ncl}(G) \subseteq U - H \subseteq F$. This proves (ii).

(ii) $\Rightarrow$ (i). Suppose $E$ and $F$ are disjoint $nI_g$-closed sets, then the $nI_g$-closed set $E$ is contained in the $nI_g$-open set $U - F$. By hypothesis, there exists a $n$-open
set \( G \) of \( U \) such that \( E \subseteq G \subseteq ncl(G) \subseteq U - F \). If \( H = U - ncl(G) \), then \( G \) and \( H \) are disjoint \( n \)-open sets containing \( E \) and \( F \) respectively. Therefore, \((U, N, I)\) is \( ngl \)-normal.

\[ \text{Theorem 3.7. In an ideal space } (U, N, I), \text{ the following are equivalent.} \]

(i) \( U \) is \( ngl \)-normal.

(ii) For each pair of disjoint \( nI_g \)-closed subsets \( E \) and \( F \) of \( U \), there exists a \( n \)-open set \( G \) of \( U \) containing \( E \) such that \( ncl(G) \cap F = \emptyset \).

(iii) For each pair of disjoint \( nI_g \)-closed subsets \( E \) and \( F \) of \( U \), there exists a \( n \)-open set \( G \) containing \( E \) and a \( n \)-open set \( H \) containing \( F \) such that \( ncl(G) \cap ncl(H) = \emptyset \).

\[ \text{Proof.} \]

(i) \( \Rightarrow \) (ii). Suppose that \( E \) and \( F \) are disjoint \( nI_g \)-closed subsets of \( U \). Then the \( nI_g \)-closed set \( E \) is contained in the \( nI_g \)-open set \( U - F \). By Theorem 3.6, there exists a \( n \)-open set \( G \) such that \( E \subseteq G \subseteq ncl(G) \subseteq U - F \). Therefore, \( G \) is the required \( n \)-open set containing \( E \) such that \( ncl(G) \cap F = \emptyset \).

(ii) \( \Rightarrow \) (iii). Let \( E \) and \( F \) be two disjoint \( nI_g \)-closed subsets of \( U \). By hypothesis, there exists a \( n \)-open set \( G \) containing \( E \) such that \( ncl(G) \cap F = \emptyset \). Also, \( ncl(G) \) and \( F \) are disjoint \( nI_g \)-closed sets of \( U \). By hypothesis, there exists a \( n \)-open set \( H \) containing \( F \) such that \( ncl(G) \cap ncl(H) = \emptyset \).

(iii) \( \Rightarrow \) (i) The proof is clear.

\[ \text{Theorem 3.8. Let } (U, N, I) \text{ be a } ngl \text{-normal space. If } E \text{ and } F \text{ are disjoint } nI_g \text{-closed subsets of } U, \text{ then there exists disjoint } n \text{-open sets } G \text{ and } H \text{ such that } ncl^*(E) \subseteq G \text{ and } ncl^*(F) \subseteq H. \]

\[ \text{Proof. Suppose that } E \text{ and } F \text{ are disjoint } nI_g \text{-closed sets. By Theorem 3.7 (iii), there exists a } n \text{-open set } G \text{ containing } E \text{ and a } n \text{-open set } H \text{ containing } F \text{ such that } ncl(G) \cap ncl(H) = \emptyset \text{. Since } E \text{ is } nI_g \text{-closed, } E \subseteq G \Rightarrow ncl^*(E) \subseteq G. \text{ Similarly } ncl^*(F) \subseteq H. \]

\[ \text{Theorem 3.9. Let } (U, N, I) \text{ be a } ngl \text{-normal space. If } E \text{ is a } nI_g \text{-closed set and } F \text{ is a } nI_g \text{-open set containing } E, \text{ then there exists a } n \text{-open set } G \text{ such that } E \subseteq ncl^*(E) \subseteq G \subseteq nint^*(F) \subseteq F. \]

\[ \text{Proof. Suppose } E \text{ is a } nI_g \text{-closed set and } F \text{ is a } nI_g \text{-open set containing } E. \text{ Since } E \text{ and } U - F \text{ are disjoint } nI_g \text{-closed sets, by Theorem 3.8, there exist disjoint} \]
n-open sets \( G \) and \( H \) such that \( ncl^*(E) \subseteq G \) and \( ncl^*(U - F) \subseteq H \). Now, \( U - nint^*(F) = ncl^*(U - F) \subseteq H \Rightarrow U - H \subseteq nint^*(F) \). Again, \( G \cap H = \emptyset \Rightarrow G \subseteq U - H \) and so \( E \subseteq ncl^*(E) \subseteq G \subseteq U - H \subseteq nint^*(E) \subseteq F \). \( \square \)

**Definition 3.4.** A subset \( E \) of an ideal nanotopological space \((U, N, I)\) is said to be a nano regular generalized closed set with respect to an ideal \( I \) (briefly \( nI_g\)-closed) if \( E^*_n \subseteq G \) whenever \( E \subseteq G \) and \( G \) is \( n\)-regular open.

\( E \) is called \( nI_g\)-open if \( U - E \) is \( nI_g\)-closed.

**Theorem 3.10.** In an ideal nanotopological space \((U, N, I)\), every \( nI_g \)-closed set is \( nI_g \)-closed.

**Proof.** Follows from the Definitions 3.1 and 3.4

**Lemma 3.1.** Let \((U, N, I)\) be an ideal nanotopological spaces. A subset \( E \subseteq U \) is \( nI_g \)-open if and only if \( P \subseteq nint^*(E) \) whenever \( P \) is \( nr \)-closed and \( P \subseteq E \).

**Proof.** Suppose that \( E \) is \( nI_g \)-open. Let \( P \) be a \( nr \)-closed set contained in \( E \).

Then \( U - E \subseteq U - P \) and \( U - P \) is \( nr \)-open. Since \( U - E \) is \( nI_g \)-closed, \( ncl^*(U - E) \subseteq U - P \) and so \( P \subseteq U - ncl^*(U - E) = nint^*(E) \).

Conversely, suppose \( U - E \subseteq G \) and \( G \) is \( nr \)-open. Then \( U - G \subseteq E \) and \( U - G \) is \( nr \)-closed. By our assumption, \( U - G \subseteq nint^*(E) \) and so \( U - nint^*(E) \subseteq G \) which implies that \( ncl^*(U - E) \subseteq G \). Therefore, \( U - E \) is \( nI_g \)-closed and so \( E \) is \( nI_g \)-open. \( \square \)

**Definition 3.5.** A space \((U, N)\) is said to be a mildly nano normal (briefly mildly \( n \)-normal), if disjoint \( nr \)-closed sets are separated by disjoint \( n \)-open sets.

**Theorem 3.11.** Let \((U, N, I)\) be an ideal nanotopological space, where \( I \) is nano completely codense. Then the following are equivalent.

(i) \( U \) is mildly \( n \)-normal.

(ii) For disjoint \( nr \)-closed sets \( E \) and \( F \), there exist disjoint \( nI_g \)-open sets \( G \) and \( H \) such that \( E \subseteq G \) and \( F \subseteq H \).

(iii) For disjoint \( nr \)-closed sets \( E \) and \( F \), there exist disjoint \( nI_g \)-open sets \( G \) and \( H \) such that \( E \subseteq G \) and \( F \subseteq H \).

(iv) For a \( nr \)-closed set \( E \) and a \( nr \)-open set \( H \) containing \( E \), there exists a \( nI_g \)-open set \( G \) of \( U \) such that \( E \subseteq G \subseteq ncl^*(G) \subseteq H \).

(v) For a \( nr \)-closed set \( E \) and a \( nr \)-open set \( H \) containing \( E \), there exists a \( n^* \)-open set \( G \) of \( U \) such that \( E \subseteq G \subseteq ncl^*(G) \subseteq H \).
(vi) For disjoint \( nr \)-closed sets \( E \) and \( F \), there exist disjoint \( n\ast \)-open sets \( G \) and \( H \), such that \( E \subseteq G \) and \( F \subseteq H \).

Proof.

(i) \( \Rightarrow \) (ii). Suppose that \( E \) and \( F \) are disjoint \( nr \)-closed sets. Since \( U \) is mildly normal, there exists \( n \)-open sets \( G \) and \( H \) such that \( E \subseteq G \) and \( F \subseteq H \). But every \( n \)-open set is a \( nI_g \)-open set. This proves (ii).

(ii) \( \Rightarrow \) (iii). The proof follows from the fact that every \( nI_g \)-open set is a \( nI_{rg} \)-open set.

(iii) \( \Rightarrow \) (iv). Suppose \( E \) is \( nr \)-closed and \( F \) is a \( nr \)-open set containing \( E \). Then \( E \) and \( U - F \) are disjoint \( nr \)-closed sets. By hypothesis, there exists disjoint \( nI_{rg} \)-open sets \( G \) and \( H \) such that \( E \subseteq G \) and \( U - F \subseteq H \). Since \( U - F \) is \( nr \)-closed and \( H \) is \( nI_{rg} \)-open, then \( U - F \subseteq nint^*(H) \) and so \( U - nint^*(H) \subseteq F \). Again, \( G \cap H = \phi \) implies that \( G \cap nint^*(H) = \phi \) and so \( ncl^*(G) \subseteq U - nint^*(H) \subseteq F \). Hence \( G \) is required \( nI_{rg} \)-open set such that \( E \subseteq G \subseteq ncl^*(G) \subseteq F \).

(iv) \( \Rightarrow \) (v). Let \( E \) be a \( nr \)-closed set and \( H \) be a \( nr \)-open set containing \( E \). Then there exists a \( nI_{rg} \)-open set \( J \) of \( U \) such that \( E \subseteq J \subseteq ncl^*(J) \subseteq H \). By Lemma 3.1, \( E \subseteq nint^*(J) \). If \( G = nint^*(J) \), then \( G \) is a \( n\ast \)-open set and \( E \subseteq G \subseteq ncl^*(G) \subseteq ncl^*(J) \subseteq H \). Therefore, \( E \subseteq G \subseteq ncl^*(G) \subseteq H \).

(v) \( \Rightarrow \) (vi). Let \( E \) and \( F \) be disjoint \( nr \)-closed subsets of \( U \). Then \( U - F \) is a \( nr \)-open set containing \( E \). By hypothesis, there exists a \( n\ast \)-open set \( G \) of \( U \) such that \( E \subseteq G \subseteq ncl^*(G) \subseteq U - F \). If \( H = U - ncl^*(G) \), then \( G \) and \( H \) are disjoint \( n\ast \)-open sets of \( U \) such that \( E \subseteq G \) and \( F \subseteq H \).

(vi) \( \Rightarrow \) (i). Let \( E \) and \( F \) be disjoint \( nr \)-closed sets of \( U \). Then there exist disjoint \( n\ast \)-open sets \( G \) and \( H \) such that \( E \subseteq G \) and \( F \subseteq H \). Since \( I \) is nano completely codense, then \( \mathcal{N}^* \subseteq \mathcal{N}^\alpha \) and so \( G, H \in \mathcal{N}^\alpha \). Hence \( E \subseteq G \subseteq nint(ncl(nint(G))) = J \) and \( F \subseteq H \subseteq nint(ncl(nint(H))) = K \). \( J \) and \( K \) are the required disjoint \( n \)-open sets containing \( E \) and \( F \) respectively. This proves (i).

\( \square \)

4. \( nI_g \)-regular spaces

Definition 4.1. An ideal nanotopological space \((U, \mathcal{N}, I)\) is said to be a nano \( I_g \)-regular space (briefly \( nI_g \)-regular space) if for each pair consisting of a point \( y \) and a closed set \( F \) not containing \( y \), there exist disjoint \( nI_g \)-open sets \( G \) and \( H \) such that \( y \in G \) and \( F \subseteq H \).
Theorem 4.1. In an ideal nanotopological space \((U, N, I)\), the following are equivalent.

(i) \(U\) is \(nI_g\)-regular.
(ii) For every \(n\)-closed set \(F\) not containing \(a\), there exist disjoint \(nI_g\)-open sets \(G\) and \(H\) such that \(a \in G\) and \(F \subseteq H\).
(iii) For every \(n\)-open set \(H\) containing \(a\), there exists a \(nI_g\)-open set \(G\) of \(U\) such that \(a \subseteq G \subseteq ncl^*(G) \subseteq H\).

Proof.
(i) and (ii) are equivalent by the Definition 4.1.

(ii) \(\Rightarrow\) (iii). Let \(H\) be a \(n\)-open subset such that \(a \in H\). Then \(U - H\) is a \(n\)-closed set not containing \(a\). Therefore, there exist disjoint \(nI_g\)-open sets \(G\) and \(S\) such that \(a \in G\) and \(U - H \subseteq S\). Now, \(U - H \subseteq S\) implies that \(U - H \subseteq nint^*(S)\) and so \(U - nint^*(S) \subseteq H\). Again, \(G \cap S = \emptyset\) implies that \(G \cap nint^*(S) = \emptyset\) and so \(ncl^*(G) \subseteq U - nint^*(S)\). Therefore, \(a \in G \subseteq ncl^*(G) \subseteq H\). This proves (iii).

(iii) \(\Rightarrow\) (i). Let \(F\) be a \(n\)-closed set not containing \(a\). By hypothesis, there exists a \(nI_g\)-open set \(G\) such that \(a \in G \subseteq ncl^*(G) \subseteq U - F\). If \(S = U - ncl^*(G)\), then \(G\) and \(S\) are disjoint \(nI_g\)-open sets such that \(a \in G\) and \(F \subseteq S\). This proves (i).

Theorem 4.2. If \((U, N, I)\) is a \(nI_g\)-regular, \(nT_1\)-space where \(I\) is nano completely codense, then \(U\) is \(nr\)-closed.

Proof. Let \(F\) be a \(n\)-closed set not containing \(a \in U\). By Theorem 4.1, there exists a \(nI_g\)-open set \(G\) of \(U\) such that \(a \in G \subseteq ncl^*(G) \subseteq U - F\). Since \(U\) is a \(nT_1\)-space, \(\{a\}\) is \(n\)-closed and so \(\{a\} \subseteq nint^*(G)\). Since \(I\) is nano completely codense, \(N^* \subseteq N^a\) and so \(nint^*(G)\) and \(U - ncl^*(G)\) are \(N^a\)-open sets. Now, \(a \in nint^*(G) \subseteq nint(ncl(nint(nint^*(G)))) = J\) and \(F \subseteq U - ncl^*(G) \subseteq nint(ncl(nint(U - ncl^*(G)))) = K\). Then \(J\) and \(K\) are disjoint \(n\)-open sets containing \(a\) and \(F\) respectively. Therefore, \(U\) is \(nr\)-closed.

Theorem 4.3. If every \(n\)-open subset of an ideal nanotopological space \((U, N, I)\) is \(n^*\)-closed, then \((U, N, I)\) is \(nI_g\)-regular.

Proof. Suppose every \(n\)-open subset of \(U\) is \(n^*\)-closed. Then every subset of \(U\) is \(nI_g\)-closed and hence every subset of \(U\) is \(nI_g\)-open. If \(F\) is a \(n\)-closed set not containing \(a\), then \(\{a\}\) and \(F\) are the required disjoint \(nI_g\)-open sets containing \(a\) and \(F\) respectively. Therefore, \((U, N, I)\) is \(nI_g\)-regular.
Theorem 4.4. Let \((U, \mathcal{N}, I)\) be an ideal topological space where \(I\) is nano completely codense. Then the following are equivalent.

(i) \(U\) is \(n_r\)-closed.

(ii) For every \(n\)-closed set \(E\) and each \(a \in U - E\), there exist disjoint \(n^*\)-open sets \(G\) and \(H\) such that \(a \in G\) and \(E \subseteq H\).

(iii) For every \(n\)-open set \(H\) of \(U\) and \(a \in H\), there exists a \(n^*\)-open set \(G\) such that \(a \in G \subseteq ncl^*(G) \subseteq H\).

Proof.

(i) \(\Rightarrow\) (ii). Let \(E\) be a \(n\)-closed subset of \(U\) and let \(a \in U - E\). Then there exist disjoint \(n\)-open sets \(G\) and \(H\) such that \(a \in G\) and \(E \subseteq H\). But every \(n\)-open set is \(n^*\)-open. This proves (ii).

(ii) \(\Rightarrow\) (iii). Let \(H\) be a \(n\)-open set containing \(a \in U\). Then \(U - H\) is \(n\)-closed and \(a \in H\). By hypothesis, there exist disjoint \(n^*\)-open sets \(G\) and \(S\) such that \(a \in G\) and \(U - H \subseteq S\). Since \(G \cap S = \emptyset\), we have \(G \subseteq U - S\) and \(U - S\) is \(n^*\)-closed. So \(ncl^*(G) \subseteq U - S \subseteq H\). Therefore, \(G\) is the required \(n^*\)-open set such that \(a \in G \subseteq ncl^*(G) \subseteq H\).

(iii) \(\Rightarrow\) (i). Let \(E\) be a \(n\)-closed set and \(a \notin E\). By (iii), there exists a \(n^*\)-open set \(G\) such that \(a \in G \subseteq ncl^*(G) \subseteq U - E\). Let \(H = U - ncl^*(G)\). Then \(E \subseteq H\), and \(G\) and \(H\) are disjoint \(n^*\)-open sets. Since \(I\) is nano completely codense, \(N^* \subseteq N^\alpha\) and so \(G\) and \(H\) are \(N^\alpha\)-open sets. Therefore, \(G \subseteq nint(ncl(nint(G))) = J\) and \(E \subseteq H \subseteq nint(ncl(nint(H))) = K\). Then \(J\) and \(K\) are disjoint \(n\)-open sets such that \(a \in J\) and \(E \subseteq K\). Hence \(U\) is \(n\)-regular.

\(\square\)

Theorem 4.5. Let \((U, \mathcal{N}, I)\) be an ideal nanotopological space, where \(I\) is nano completely codense. Then the following are equivalent.

(i) \(U\) is almost \(n\)-regular.

(ii) For each \(n_r\)-closed set \(E\) and each \(a \in U - E\), there exist disjoint \(n^*\)-open sets \(G\) and \(H\) such that \(a \in G\) and \(E \subseteq H\).

(iii) For each \(n_r\)-open set \(H\) and \(a \in H\), there exists a \(n^*\)-open set \(G\) such that \(a \in G \subseteq ncl^*(G) \subseteq H\).

Proof.
(i) ⇒ (ii). Let $E \subseteq U$ be $nr$-closed and $a \in U - E$. Then there exist disjoint $n$-open sets $G$ and $H$ such that $a \in G$ and $E \subseteq H$. Since every $n$-open set is a $n^\star$-open set, the proof follows.

(ii) ⇒ (iii). Let $H$ be $nr$-open and $a \in H$. By (ii), there exist disjoint $n^\star$-open sets $G$ and $S$ such that $a \in G$ and $U - H \subseteq S$. Since $G \cap S = \phi$, we have $ncl^\ast(G) \subseteq U - S \subseteq H$. Therefore, $G$ is the required $n^\star$-open set such that $a \in G \subseteq ncl^\ast(G) \subseteq H$.

(iii) ⇒ (i). Let $E$ be $nr$-closed and $a \in U - E$. By hypothesis, there exists a $n^\star$-open set $G$ such that $a \in G \subseteq ncl^\ast(G) \subseteq U - E$. Let $H = U - ncl^\ast(G)$. Then $E \subseteq H$, and $G$, $H$ are disjoint $n^\star$-open sets. Since $I$ is nano completely codense, $n\tau^\star \subseteq n\tau^\alpha$ and so $G$ and $H$ are $na$-open sets. Therefore, we have $a \in G \subseteq nint(ncl(nint(G))) = J$ and $E \subseteq H \subseteq nint(ncl(nint(H))) = K$. Then $J$ and $K$ are the required disjoint $n$-open sets such that $a \in J$ and $E \subseteq K$. Hence $U$ is almost $n$-regular.

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References


Department of Mathematics,
Senthamarai College of Arts and Science,
Madurai District, Tamil Nadu, India.
Email address: prem.rpk27@gmail.com

Department of Mathematics,
Pasumpon Muthuramalinga Thevar College, Usilampatti,
Madurai District, Tamil Nadu, India.
Email address: proframesh9@gmail.com

Department of Mathematics,
Ananda College,
Sivaganga District, Tamil Nadu, India.
Email address: sanantodavid@gmail.com