SOME ALGEBRAIC STRUCTURES ON MAX-MAX, MIN-MIN COMPOSITIONS OVER INTUITIONISTIC FUZZY MATRICES

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ABSTRACT. In this article, some algebraic properties of two composition operators max-max ($\lor_m$) and min-min ($\land_m$) are studied on Intuitionistic fuzzy matrices. Also some comparisons of these new composition operator with other well known like max-min ($\lor$) and min-max ($\land$) are investigated. Finally some algebraic structures are constructed using the above said operators over the set of all intuitionistic fuzzy matrices.

1. INTRODUCTION

The concept of fuzzy set has been found to be an effective tool to deal with fuzziness. However it often falls short of the expected standard in the description of neutral state. As a result a new concept called IFS was introduced by Atanassov in [1, 2]. Im et.al. in [5, 6] and Khan et al. in [8] generalizes fuzzy matrix as IFM i with its operations and has been useful in dealing with the areas such as decision making, relational equations, clustering analysis etc. IFM product through max-min composition is investigated by Khan and Pal in [9]. Ragab and Emam defined min-max operation in [10]. Murugadas and Sriram constructed a semiring structure in IFM theory from the operations max-min IFM product and component wise max-min operation in [12]. Later Emam and Fndh extended component wise min-max operation into min-max IFM product.
in [4] and discussed its algebraic properties. Muthuraji et al. introduced component wise min-min operation and studied some algebraic properties with an IFM decomposition in [13]. Max-max IFM product is introduced in [11] by Riyaz Ahmed Padder and Murugadas and they proved this operation is more relevant than max-min IFM product. In this way min-min IFM product is established by Lalitha in [7]. In this study various algebraic properties of max-max and min-min compositions are discussed which gives some algebraic structures on IFMs.

2. Preliminaries

2.1. Intuitionistic Fuzzy Set and its Operations. In [1, 2] Atanassov defined an IFS \( A \) in \( E \) (Universal Set) is defined as an object of the following form
\[
A = \{(x, \mu_A(x_i), \gamma_A(x_i)) \mid x \in E\},
\]
where the functions \( \mu_A(x) : E \rightarrow [0, 1] \) and \( \gamma_A(x) : E \rightarrow [0, 1] \) define the membership and non-membership function of the element \( x \in E \) respectively for every \( x \in E \),

\[0 \leq \mu_A(x) + \gamma_A(x) \leq 1.\]

For our convenience consider the elements of IFSs as in the form \((x, x')\):

(i) Component wise max-min operation \((x, x') \lor (y, y') = [\max(x, y), \min(x', y')]\)

(ii) Component wise min-max operation \((x, x') \land (y, y') = [\min(x, y), \max(x', y')]\)

(iii) \((x, x')' = (x', x)\)

2.2. Intuitionistic Fuzzy Matrix and its Operations. Khan et al. [8, 9] defined IFM and operations as follows Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of alternatives and \( Y = \{y_1, y_2, \ldots, y_n\} \) be the attribute set of each element of \( X \). An IFM is defined by
\[
A = \left\{ (x_i, y_j, \mu_A(x_i, y_j), \gamma_A(x_i, y_j)) \mid i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \right\}
\]
where \( \mu_A : X \times Y \rightarrow [0, 1] \) and \( \gamma_A : X \times Y \rightarrow [0, 1] \) satisfy the condition
\[0 \leq \mu_A(x_i, y_j) + \gamma_A(x_i, y_j) \leq 1.\]

For simplicity we denote an IFM as matrix of pairs \( A = [(a_{ij}, a'_{ij})] \) of non-.. real numbers satisfying \( a_{ij} + a'_{ij} \leq 1 \) for all \( i, j \). If \( A, B \in \mathcal{F}_{mn} \), set of all IFMs of order \( m \times n \) then

(i) \( A \lor B = [\max(a_{ij}, b_{ij}), \min(a'_{ij}, b'_{ij})] \) for all \( i, j \) (Component wise max-min operation)

(ii) \( A \land B = [\min(a_{ij}, b_{ij}), \max(a'_{ij}, b'_{ij})] \) for all \( i, j \) (Component wise min-max operation)

If \( A \in \mathcal{F}_{mn}, B \in \mathcal{F}_{np} \) then the max-min and min-max IFM product is defined by
(iii) \( A \lor B = \left[ \bigvee_{K=1}^{p} (a_{ik} \land b_{kj}), \bigwedge_{K=1}^{p} (a'_{ik} \lor b'_{kj}) \right] \) for all \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, p \)

(iv) \( A \land B = \left[ \bigvee_{K=1}^{p} (a_{ik} \lor b_{kj}), \bigwedge_{K=1}^{p} (a'_{ik} \land b'_{kj}) \right] \)

(v) If \( A = (a_{ij}, a'_{ij}) = (1, 0) \) for all \( i, j \) then \( A \) is said to be an Universal matrix denoted by \( U \).

(vi) If \( A = (a_{ij}, a'_{ij}) = (0, 1) \) for all \( i, j \) than \( A \) is said to be Zero matrix denoted by \( 0 \).

(vii) If \( A = \begin{cases} (1, 0) & \text{if } i = j \\ (0, 1) & \text{if } i \neq j \end{cases} \), then \( A \) is said to be identity matrix denoted by \( I_n \).

2.3. Max-max and Min-min IFM product. Riyas Ahmed Padder and Murugadas in [11] defined max-max product of two square IFMs as follows \( A, B \in \mathcal{F}_{n \times n} \) with all comparable entries is defined as

\[
A \circ B = \left[ \left( \bigvee_{K=1}^{n} (a_{iK} \lor b_{Kj}), \bigwedge_{K=1}^{n} (a'_{ik} \land b'_{kj}) \right) \right].
\]

The min-min product of two square IFMs is defined by Lalitha in [7] \( A, B \in \mathcal{F}_{n \times n} \) is defined as

\[
A \cdot B = \left[ \left( \bigwedge_{K=1}^{n} (a_{iK} \land b_{Kj}), \bigvee_{K=1}^{n} (a'_{ik} \lor b'_{kj}) \right) \right]
\]

for our convenience, we consider the two operations in different notations as \( \lor_m \) and \( \land_m \) as after.

3. ALGEBRAIC PROPERTIES OF \( \lor_m \) AND \( \land_m \)

The max-max product \( (\lor_m) \) of two rectangular IFMS \( A \in \mathcal{F}_{m \times n} \) and \( B \in \mathcal{F}_{n \times p} \) is defined by

\[
A \lor_m B = \left[ \left( \bigvee_{K=1}^{n} (a_{iK} \lor b_{Kj}), \bigwedge_{K=1}^{n} (a'_{ik} \land b'_{kj}) \right) \right],
\]

where \( i \) varies from 1 to \( m, j \) from 1 to \( p \) and \( k \) from 1 to \( n \):

\[
A \land_m B = \left[ \left( \bigwedge_{K=1}^{n} (a_{iK} \land b_{Kj}), \bigvee_{K=1}^{n} (a'_{ik} \lor b'_{kj}) \right) \right].
\]
Hence it is not necessary the elements of $A$ and $B$ are comparable.

**Lemma 3.1.** For any $A \in \mathcal{F}_{n \times n}$, we have the following

(i) $A \lor_m U = U$.
(ii) $A \land_m 0 = 0$.
(iii) $A \lor_m I_n = U$.
(iv) $A \land_m I_n = 0$.
(v) $A \lor_m A \geq A$ gives $\lor_m$ is reflexive.
(vi) $A \land_m A \leq A$ gives $\land_m$ is transitive.

**Proof.**

(i) Let $A = [(a_{ij}, a'_{ij})]$ and $U = [(1, 0)]$ for all $i, j$ the $(i, j)^{th}$ element of

$$A \lor_m U = \left[\left( \bigvee_{K=1}^{n} (a_{iK} \lor 1), \bigwedge_{K=1}^{n} (a'_{ik} \land 0) \right) \right] = [(1, 0)] = U.$$

(v) Consider the $(i, k)^{th}$ element of $A \lor_m A$ as

$$= \left[\left( \bigvee_{K=1}^{n} (a_{iK} \lor a_{Kj}), \bigwedge_{K=1}^{n} (a'_{ik} \land a'_{kj}) \right) \right].$$

It is clear $\bigvee_{K=1}^{n} (a_{iK} \lor a_{Kj}) \geq a_{ij}$ and $\bigwedge_{K=1}^{n} (a_{iK} \land a'_{Kj}) \leq a_{ij}$

$\therefore A \lor_m A \geq A$.

Similar way we can prove the other results also. \hfill \square

**Lemma 3.2.** For any two IFMs $A, B \in \mathcal{F}_{n \times n}$ we have the following

(i) $A \lor_m B = U$ if either $A$ or $B$ are reflexive.
(ii) $A \land_m B = 0$ if either $A$ or $B$ are irreflexive.
(iii) $A \lor_m B \neq B \lor_m A$ which means $\lor_m$ is not commutative.
(iv) $A \land_m B \neq B \land_m A$ which means $\land_m$ is not commutative.

**Proof.**

(i) Consider the $(i, k)^{th}$ element of $A \lor_m B$ is

$$\left[\left( \bigvee_{K=1}^{n} (a_{ik} \lor b_{kj}), \bigwedge_{K=1}^{n} (a'_{ik} \land b'_{kj}) \right) \right].$$

If $A$ is reflexive then $(a_{ii}, a'_{ii}) = (1, 0)$ for all $i, j$ when $k = i$, $a_{ii} \lor b_{ij} = 1$ and $a'_{ii} \land b'_{ij} = 0$ and $\bigvee_{K=1}^{n} (a_{ik} \lor b_{kj}) = 1$ and $\bigwedge_{K=1}^{n} (a'_{ik} \land b'_{kj}) = 0$ for all $k = i$. $\therefore A \lor_m B = U$.

(ii) Similar to (i).
(iii) Since \((a_{ik} \lor m b_{kj}) \neq (b_{ik} \lor m a_{kj})\) and \((a_{ik} \land m b_{kj}) \neq (b_{ik} \land m a_{kj})\) for all \(i, j, k\).
\[\therefore \lor m \text{ and } \land m \text{ are not commutative.} \]

**Lemma 3.3.** For any three IFMs \(A \in \mathcal{F}_{m \times n}, B \in \mathcal{F}_{n \times p}\) and \(C \in \mathcal{F}_{p \times q}\), we have

(i) \((A \lor m B) \lor m C = A \lor m (B \lor m C)\),

(ii) \((A \land m B) \land m C = A \land m (B \land m C)\),

which gives the operations \(\lor m\) and \(\land m\) satisfy associative property on the set of all IFMs.

**Proof.** Let \(A = [(a_{ij}, a'_{ij})], B = [(b_{jk}, b'_{jk})]\) and \(C = [(c_{kl}, c'_{kl})]\), where \(i = 1, 2, \ldots, m\) and \(j = 1, 2, \ldots, n\) and \(k = 1, 2, \ldots, p\).

Now consider the \((i, k)\)th element of the product

\[A \lor m B = \left[ \bigvee_{j=1}^{n} (a_{ij} \lor b_{jk}), \bigwedge_{j=1}^{n} (a'_{ij} \land b'_{jk}) \right].\]

In this way the \((i, l)\)th element of the product

\[(A \lor m B) \lor m C = \left[ \bigvee_{k=1}^{p} \left\{ \bigvee_{j=1}^{n} (a_{ij} \lor b_{jk}), \bigwedge_{j=1}^{n} (a'_{ij} \land b'_{jk}) \right\} \lor c_{kl}, \bigwedge_{k=1}^{p} \left\{ \bigwedge_{j=1}^{n} (a'_{ij} \land b'_{jk}) \land c'_{kl} \right\} \right].\]

Similarly the \((j, l)\)th element of \(B \lor m C\)

\[= \left[ \bigvee_{k=1}^{p} (b_{jk} \lor c_{kl}), \bigwedge_{k=1}^{p} (b'_{jk} \land c'_{kl}) \right].\]

Also the \((i, l)\)th element of \(A \lor m (B \lor m C)\)

\[= \left[ \bigvee_{j=1}^{n} a_{ij} \left\{ \bigvee_{k=1}^{p} (b_{jk} \lor c_{kl}), \bigwedge_{k=1}^{p} (b'_{jk} \land c'_{kl}) \right\} \right], \bigwedge_{j=1}^{n} a'_{ij} \left\{ \bigwedge_{k=1}^{p} (b_{jk} \land c_{kl}), \bigwedge_{k=1}^{p} (b'_{jk} \land c'_{kl}) \right\}.\]

from the above two equations, we have

\[(A \lor m B) \lor m C = A \lor m (B \lor m C).\]

(ii) similar to (i).
Lemma 3.4. For any three IFMs $A, B$ and $C$ the operations $\wedge_m$ and $\wedge_m$ are not distributive to each other that is

(i) $A \wedge_m (B \vee_m C) \neq (A \wedge_m B) \vee_m (A \wedge_m C)$.
(ii) $A \vee_m (B \wedge_m C) \neq (A \vee_m B) \wedge_m (A \vee_m C)$.
(iii) $A \wedge (B \wedge_m C) \neq (A \wedge B) \wedge_m (A \wedge C)[\wedge$ is not left distributive with $\wedge_m]$.
(iv) $(A \wedge_m B) \wedge C \neq (A \wedge C) \wedge_m (B \wedge C)[\wedge$ is not right distributive with $\wedge_m]$.

The above results can be illustrated by the following example. Consider any $2 \times 2$ IFMs $A, B, C$ as

\[ A = \begin{bmatrix} .2 & .4 \\ .4 & 1 \end{bmatrix}, B = \begin{bmatrix} .01 & .5 \\ .5 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} .3 & .4 \\ .4 & 1 \end{bmatrix}. \]

Also $A \wedge_m B = \begin{bmatrix} .01 & .5 \\ .5 & 1 \end{bmatrix}$ and $A \wedge_m C = \begin{bmatrix} .3 & .4 \\ .3 & 1 \end{bmatrix}$, gives $(A \wedge_m B) \vee_m (A \wedge_m C) = \begin{bmatrix} .3 & 4 \\ .3 & 1 \end{bmatrix}$.

Thus

\[ A \wedge_m (B \vee_m C) \neq (A \wedge_m B) \vee_m (A \wedge_m C). \]

In this way we can prove

\[ A \vee_m (B \wedge_m C) \neq (A \vee_m B) \wedge_m (A \vee_m C). \]

Lemma 3.5. For any three square IFMs $A, B$ and $C$, we have

(i) $A \wedge_m (B \wedge C) = (A \wedge_m B) \wedge (A \wedge_m C)$ [\wedge_m is left distributive with component wise min-max operation $\wedge$];
(ii) $(A \wedge B) \wedge_m C = [A \wedge_m C] \wedge [B \wedge_m C]$ [\wedge_m is right distributive over $\wedge$];
(iii) $A \vee_m (B \vee C) = (A \vee_m B) \vee (A \vee_m C)$ [\vee_m is left distributive over $\vee$];
(iv) $(A \vee B) \vee_m C = (A \vee_m C) \vee (B \vee_m C)$ [\vee_m is right distributive over $\vee$].

Proof. Let $A = [(a_{ij}, a'_{ij})], B = [(b_{ij}, b'_{ij})]$ and $C = [(c_{ij}, c'_{ij})]$. Consider the $(i, j)^{th}$ element of $B \wedge C = [(b_{ij} \wedge c_{ij}), (b'_{ij} \vee c'_{ij})]$. Now

\[
A \wedge_m (B \wedge C) = \wedge_k^n (a_{ik} \wedge b_{kj}), \bigvee_{k=1}^n (a'_{ik} \vee b'_{kj}) \text{ if } b_{ij} < c_{ij} \text{ and } b'_{ij} > c'_{ij}
\]

\[ = \wedge_k^n (a_{ik} \wedge c_{kj}), \bigvee_{k=1}^n (a'_{ik} \vee c'_{kj}) \text{ if } b_{ij} > c_{ij} \text{ and } b'_{ij} < c'_{ij}. \]
\[
\forall_{k=1}^{n}(a_{ik} \land b_{kj}), \bigvee_{k=1}^{n}(a'_{ik} \lor c'_{kj}) \text{ if } b_{ij} < c_{ij} \text{ and } b'_{ij} < c'_{ij}
\]

\[
\forall_{k=1}^{n}(a_{ik} \land c_{kj}), \bigvee_{k=1}^{n}(a'_{ik} \lor b'_{kj}) \text{ if } b_{ij} > c_{ij} \text{ and } b'_{ij} > c'_{ij}
\]

\[
A \land_m B = [\bigwedge_{k=1}^{n}(a_{ik} \land b_{kj}), \bigvee_{k=1}^{n}(a'_{ik} \lor b'_{kj})]
\]

\[
A \land_m C = [\bigwedge_{k=1}^{n}(a_{ik} \land c_{kj}), \bigvee_{k=1}^{n}(a'_{ik} \lor b'_{kj})]
\]

\[
(A \land_m B) \land (A \land_m C) = \bigwedge_{k=1}^{n}(a_{ik} \land b_{kj}), \bigvee_{k=1}^{n}(a'_{ik} \lor b'_{kj}) \text{ if } b_{ij} < c_{ij} \text{ and } b'_{ij} > c'_{ij}
\]

\[
\vdots. \land_m \text{ is left distributive with } \land.
\]

In this way, we can prove the other results. \hfill \square

**Lemma 3.6.** The operations \( \land_m \) and \( \lor_m \) satisfy demorgan’s law on the set of IFMs

(i) \( (A \lor_m B)^C = A^C \land_m B^C \)

(ii) \( (A \land_m B)^C = A^C \lor_m B^C \)

**Proof.**

\[
[A \lor_m B]^C = \left[\bigwedge_{k=1}^{n}(a'_{ik} \land b'_{kj}), \bigvee_{k=1}^{n}(a_{ik} \lor b_{kj})\right]
\]

\[
A^C = [(a'_{ij}, a_{ij})] \text{ and } B^C = [(b'_{ij}, b_{ij})]
\]

\[
A^C \land_m B^C = \left[\bigwedge_{k=1}^{n}(a'_{ik} \land b'_{kj}), \bigvee_{k=1}^{n}(a_{ik} \lor b_{kj})\right]
\]

from (1) and (2)

\[
(A \lor_m B)^C = A^C \land_m B^C.
\]

(ii) similar to (i). \hfill \square

4. ALGEBRAIC STRUCTURES ON \( \lor_m \) AND \( \land_m \)

**Theorem 4.1.** If \( S_1 \) denote the set of all rectangular IFMs then we have

(i) \( (S_1, \lor_m) \) is a semigroup.

(ii) \( (S_1, \land_m) \) is a semigroup.

**Proof.** Let \( A \) and \( B \) are two IFMs of order \( m \times n \) and \( n \times p \). Since \( A \lor_m B \in S_1 \) and \( A \land_m B \in S_1, \land_m \) and \( \lor_m \) are closed. From Lemma 3.3. \( \land_m \) and \( \lor_m \) are
associative.
\[ (S_1, \wedge_m) \text{ and } (S_1, \vee_m) \text{ form a semigroup.} \]

**Theorem 4.2.** If \( S_2 \) denote the set of all square IFMs then we have the following structures

\( (i) \ (S_2, \wedge, \wedge_m) \text{ is a semiring.} \)

\( (ii) \ (S_2, \vee, \vee_m) \text{ is a semiring.} \)

**Proof.** Consider any three square IFMs \( A, B \) and \( C \) in \( S_2 \). It is clear that the component wise max-min and min-max operators \( \vee \) and \( \wedge \) we commutative and associative. Also we have \( A \vee O = A \) and \( A \wedge U = A \) means the zero matrix and unitary matrix acts as a identity of \( \vee \) and \( \wedge \). From Lemma 3.5 \( \vee_m \) and \( \wedge_m \) are left and right distributive with component wise max-min operator \( \vee \) and min-max operator \( \wedge \).

\[ (S_2, \wedge, \wedge_m) \text{ and } (S_2, \vee, \vee_m) \text{ are semirings over the set of all IFMs with square order.} \]

**Remark 4.1.**

\( (i) \) In general the \((i, j)^{th}\) entry of max-max product of any two rectangular or square IFM give the maximum membership and minimum non-membership value of \( i^{th} \) row and \( k^{th} \) column respectively.

\( (ii) \) Similar way the \((i, k)^{th}\) entry of min-min product of any two rectangular or square IFM give the minimum membership and maximum non-membership value of \( i^{th} \) row and \( k^{th} \) column respectively.

5. **Conclusion**

Some properties of max-max and min-min IFM products are studied. Various algebraic structures are constructed from these max-max and min-min IFM products and also with other well known component wise max-min and min-max operations. All the IFM theory application using the predefined operations are going to study under these new operations in future.

**References**


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