REVERSE WIENER INDEX OF UNITARY ADDITION CAYLEY GRAPHS

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Abstract. In this paper Reverse Wiener Index of Unitary Addition Cayley graph \(G_n\) is calculated. The vertices of Unitary Addition Cayley graph \(G_n\) is \(Z_n = 0, 1, ..., (n - 1)\). If \(\gcd(x + y, n) = 1\), then the vertices \(x\) and \(y\) are adjacent.

1. Introduction

Let \(G\) be a connected graph with vertex and edge sets \(V(G)\) and \(E(G)\) respectively. The distance between the vertices \(v_i\) and \(v_j\) of \(G\) is defined as the number of edges in a minimal path connecting the vertices \(v_i\) and \(v_j\) and is denoted by \(d(v_i, v_j)\).

"For a positive integer \(n > 1\), the Unitary Addition Cayley graph \(G_n\) is the graph whose vertex set is \(Z_n\), the integers modulo \(n\) and if \(U_n\) be the set of all units of the ring \(Z_n\), then two vertices \(x, y\) are adjacent if and only if \(x + y \in U_n\)" [3].

The topological index is a numerical parameter of a graph, and is invariant under graph automorphism. The Wiener Index was introduced by Harry Wiener in [2,4]. Let \(W(G)\) be the Wiener Index of \(G\) and is defined as the sum of the distances between all unordered pairs of vertices. i.e., \(W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(v_i, v_j)\) [1].

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The largest distance between any two vertices \(v_i\) and \(v_j\) of the graph \(G\) is the diameter \(d\) of the graph. In [1,5] the Reverse Wiener matrix

\[
[RW]_{ij} = \begin{cases} 
    d - d(v_i, v_j) & \text{if } i \neq j \\
    0 & \text{if } i = j
\end{cases}
\]

The Reverse Wiener Index is \(RW(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [RW]_{ij}\).

**Theorem 1.1.** ([5]) The Wiener Index of Unitary Addition Cayley graph \(G_n\) is as follows

\[
W(G_n) = \begin{cases} 
    \frac{n^2 - 1}{2} & n \text{ is a prime number} \\
    \frac{3}{4} n^2 - 4 & n = 2^\alpha, \alpha > 1 \\
    \frac{5}{4} n^2 - n\phi(n) - n & n \text{ is even and prime divisor} \\
    (n - 1)(n - \frac{\phi(n)}{2}) & n \text{ odd but not prime}
\end{cases}
\]

where \(\phi(n)\) denotes the Euler pi function.

1.1. Reverse Wiener Index Of Unitary Addition Cayley Graph.

**Theorem 1.2.** If \(G_n\) is the Unitary Addition Cayley graph on \(n\) vertices \((n \geq 2)\) then the Reverse Wiener Index of \(G_n\) is as follows:

\[
RW(G_n) = \begin{cases} 
    \left(\frac{n}{2}\right)^2 & n = 2^\gamma, \gamma > 1 \\
    \frac{(n-1)^2}{2} & n \text{ is prime} \\
    \frac{n^2 - 2n + 4n\phi(n)}{4} & n \text{ is even and ha prime divisor} \\
    \frac{(n-1)\phi(n)}{2} & n \text{ odd not a prime number}
\end{cases}
\]

**Proof.** Consider Unitary Addition Cayley graph \(G_n\). To prove this theorem in four classifications of \(n\).

**Case (i):** When \(n = 2^\gamma, \gamma > 1\), \(G_n\) is a complete bipartite graph with bipartition of set \(V(G_n)\) into \(X = \{0, 2, ..., (n - 2)\}\) and \(Y = \{1, 3, ..., (n - 1)\}\). Hence \(d(u,v) = 1\) or \(2\) \(u, v \in V(G_n)\). Therefore diameter of \(d\) is \(2\), \(d(u,v) = 1\) is \(\frac{n^2}{4}\) times and \(d(u, v) = 2\) is \(\frac{n(n-2)}{4}\) times. Hence Reverse Wiener Index \(RW(G_n) = \frac{n^2}{4}\).

**Case (ii):** Let \(n\) be prime. The distance between any two vertices are 1 or 2. Therefore, diameter is 2. Here \(\frac{n-1}{2}\) vertices have distance 2 and \(\frac{(n-1)^2}{2}\) vertices
have distance 1. By the definition of Reverse Wiener Index consider the distance 1 as \( \frac{(n-1)^2}{2} \). Therefore \( RW(G_n) = \frac{(n-1)^2}{2} \).

**Case (iii):** Let \( n \) be an even number and has an odd prime divisor, where \( n \neq 2^\gamma, \gamma > 1 \). \( G_n \) is a bipartite graph with vertex set \( V(G_n) \) as \( X \cup Y \). \( d(u,v) \) is either \( u \in X \) or \( u \in Y \).

Suppose \( u \in X \) and \( v \in X \) clearly \( u \) and \( v \) are not adjacent. Therefore \( d(u,v)=2 \) occurs in \( \frac{(n^2-2n)}{4} \) times.

Suppose \( u \in X \) and \( v \in Y \). Now we take \( Y = S \cup T \) where \( S = \{ v \in Y ; uv \in E(G_n) \} \) and \( T = \{ v \in Y ; uv \notin E(G_n) \} \). Clearly \( d(u,v)=1 \) is \( \frac{n\phi(n)}{2} \) times.

Let \( v \in T \), \( u \) and \( v \) are not adjacent, hence \( d(u,v)=3 \) occurs in \( \frac{n^2-2n\phi(n)}{4} \) times.

Therefore, diameter is 3. Hence \( RW(G_n) = \frac{n^2-2n+4n\phi(n)}{4} \).

**Case (iv):** Let \( n \) be odd but not prime. Let \( p_1, p_2, ..., p_t \) be the different prime divisors of \( n \), \( n = p_1^{r_1} \times p_2^{r_2} \times ... \times p_t^{r_t} \) and \( p_i \neq 2, 1 \leq i \leq t \). Hence \( d(u,v)=1 \) or 2. Therefore, diameter is 2, here \( \frac{(n-1)}{2} \phi(n) \) vertices have distance 1 and \( \frac{(n-1)}{2} (n - \phi(n)) \) vertices have distance 2.

Therefore Reverse Wiener Index \( RW(G_n) = \frac{(n-1)\phi(n)}{2} \). \( \square \)

1.2. **Relation between Wiener Index and Reverse Wiener Index.** Let \( G \) be a graph with \( n \) vertices, diameter \( d \), Wiener Index \( W(G) \) and Reverse Wiener Index \( RW(G) \). Then \( RW(G) = \frac{1}{2} n(n-1) d - W(G) \).

**REFERENCES**


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