SOME NOTES ON CUBIC SPLINE OF PERIODIC FUNCTION

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ABSTRACT. A spline function is a piecewise polynomial function of order $m$ joined smoothly so that it has $m - 1$ continuous derivatives. In [1] interpolation of cubic spline function is discussed and this paper extends the results on third order spline of periodic function.

1. INTRODUCTION

In approximation theory spline interpolation, as a part of osculatory interpolation, is a form of interpolation where the interpolant is an adequate piecewise function which represents spline function. This kind of interpolation was introduced by I. J. Schoenberg in 1946 [5].

The spline function avoids the discontinuities in slope that occur with ordinary piecewise functions, except that in the $n$-th derivative where there is flexibility with respect to continuity.

The aim of this paper is to present for periodic functions belonging to $C^2[0, 2\pi]$ the analogues of the recent developments on cubic spline functions and their role in approximation theory.

Initially, we will give some concepts, definitions and notations.

Definition 1.1. Let $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a subdivision of the segment $[a, b]$. A function $S^m_\Delta : [a, b] \to \mathbb{R}$, $m \in \mathbb{N}$ is called a spline of order $m$.

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with respect to the subdivision $\Delta$ if $S^m_{f\Delta} \in C^{m-1}$, in other words it has continuous derivatives up to order $m - 1$ in $[a, b]$ and reduces to a polynomial of order smaller or equal to $m$ in each of the intervals $(-\infty, x_1), [x_1, x_2), \ldots, [x_n, \infty)$.

By $S^m_n$ we denote all splines of order $m$ for a fixed subdivision of segment in $n$ pieces. $S^m_n$ is linear space of dimension $m + n$.

**Lemma 1.1.** Let $f(x) \in C^2[a, b]$. For each subdivision of the segment $[a, b]$, 

$$\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

there exists one and only one spline function in $S^3_n$ denoted by $S^3_{f\Delta}$, such that

$$S^3_{f\Delta}(x_i) = f(x_i), i = 0, n$$

and

$$\left(S^3_{f\Delta}\right)''(x_0) = 0 = \left(S^3_{f\Delta}\right)''(x_n).$$

**Definition 1.2.** The inner product

$$\langle f, g \rangle_n = \int_0^{2\pi} f^{(n)}(t)g^{(n)}(t)dt$$

is defined for functions $f$ and $g$ which have a square-integrable $n$-th derivative on segment $[0, 2\pi]$. We define the pseudo-norm $\|f\|_n = \sqrt{\langle f, f \rangle_n}$ on linear spaces $C^n[0, 2\pi]$ where $\|f\|_n = 0$ iff $f(t), t \in [0, 2\pi]$ is polynomial up to order $n - 1$ [2].

2. SOME EXAMPLES OF SPLINES

**Example 1.** $B^0$—splines. These functions are splines of order 1 and $B^1$—splines: these functions are splines of order 1 and reach a peak at $x = x_{i+1}$ and is upward (downward) sloping for $x < x_{i+1}(x > x_{i+1})$ [3].

$$B^0_i = \begin{cases} 1 & x_i < x < x_{i+1} \\ 0 & \text{elsewhere} \end{cases}$$

and

$$B^1_i = \begin{cases} \frac{x-x_i}{x_{i+1}-x_i} & x_i < x < x_{i+1} \\ \frac{x_{i+1}-x}{x_{i+2}-x} & x_{i+1} < x < x_{i+2} \\ 0 & \text{elsewhere} \end{cases}$$
Example 2. Higher order spline functions are defined by the recursion:
\[ B^n_i(x) = \left( \frac{x - x_i}{x_{i+1} - x_i} \right) B^{n-1}_i(x) + \left( \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}} \right) B^{n-1}_{i+1}(x). \]

Example 3. Cubic splines. These functions are splines of order three and its analytic expression is
\[ S^3_{f\Delta} = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \]
for \( x \in [x_i, x_{i+1}] \) where
\[ a_i = f(x_i) = f_i, b_i = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{h_{i+1}}{6} (2\psi_i + \psi_{i+1}), c_i = \frac{\psi_i}{2}, d_i = \frac{\psi_{i+1} - \psi_i}{6h_{i+1}} \]
\[ \psi_i = \left( S^3_{f\Delta} \right)^{\prime\prime}(x_i) \] are solutions of system of linear equations
\[ \begin{align*}
\mu_1 \psi_0 + 2\psi_1 + \nu_1 \psi_2 &= \lambda_1 \\
\mu_2 \psi_1 + 2\psi_2 + \nu_2 \psi_3 &= \lambda_2 \\
& \vdots \\
\mu_{n-1} \psi_{n-2} + 2\psi_{n-1} + \nu_{n-1} \psi_n &= \lambda_{n-1}
\end{align*} \]
as amended by the given boundary conditions,
\[ h_i = x_i - x_{i-1}, \mu_i = \frac{h_i}{h_i + h_{i+1}}, \nu_i = 1 - \mu_i, \lambda_i = 6f[x_{i-1}, x_i, x_{i+1}], \]
where \( f[x_{i-1}, x_i, x_{i+1}] \) is second order divided difference.

Example 4. A cubic periodic spline \( S^3_{f\Delta} \) on segment \([0, 2\pi]\) segment is a spline function of order three such that \( \left( S^3_{f\Delta} \right)^{(k)}(0) = \left( S^3_{f\Delta} \right)^{(k)}(2\pi), k = 0, 1, 2 \), where \( \Delta = \{0 = x_0 < x_1 < \ldots < x_n = 2\pi\} \) and \( f(x) \in C^2[0, 2\pi] \) is \( 2\pi \)-periodic [4].

Note: The interpolating function \( S^3_{f\Delta} \) minimizes the value \( \int_0^{2\pi} (g''(t))^2 dt \) among all functions \( g \in C^2 \) which coincide with function \( f(x) \) at the points \( x_i, i = 0, n \).

3. Main Results

Theorem 3.1. If \( f(x) \in C^2[0, 2\pi] \) is \( 2\pi \)-periodic then for some \( c \) intermediate to \( x_{i-1}, x_i \) and \( x_{i+1} \)
\[ -4.5 - \frac{(f''(c))^2}{2} \leq (S^3_{f\Delta})''(x_i) \leq 4.5 + \frac{(f''(c))^2}{2}, i = 1, n - 1, \]
where \( \Delta = \{0 = x_0 < x_1 < \ldots < x_n = 2\pi\} \).
Proof. Let us take $\psi_k = \max_{1 \leq i \leq n-1} |\psi_i|$. Then using Example 1 we get

$$\max_{1 \leq i \leq n-1} |\lambda_i| \geq |\lambda_k| \geq 2|\psi_k| - \mu_k|\psi_{k-1}| - \nu_k|\psi_{k+1}| \geq 2|\psi_k| - \mu_k|\psi_k| - \nu_k|\psi_k| = |\psi_k|(2 - (\mu_k + \nu_k)) = |\psi_k|(2 - 1) = |\psi_k| = \max_{1 \leq i \leq n-1} |\lambda_i|.$$  

Consequently

$$\max_{1 \leq i \leq n-1} \left| \left( S^3_{f\Delta} \right)''(x_i) \right| \leq \max_{1 \leq i \leq n-1} |\lambda_i|.$$  

Now from obtained result, the Arithmetic-Geometric inequality and the Mean-Value theorem as applied to the second order difference assure that

$$\max_{1 \leq i \leq n-1} \left| \left( S^3_{f\Delta} \right)''(x_i) \right| \leq \frac{9 + (f''(c))^2}{2}, i = 1, n - 1,$$

where $c$ intermediate to $x_{i-1}$, $x_i$ and $x_{i+1}$.

Consequently

$$\left| \left( S^3_{f\Delta} \right)''(x_i) \right| \leq \frac{9 + (f''(c))^2}{2}, i = 1, n - 1,$$

which yields to our result.

\[\square\]

Theorem 3.2. If the Fourier series of periodic function $f(x) \in C^2[-\pi, \pi]$ contains only cosine terms, then the Fourier series of the interpolating spline $S^3_{f\Delta}$, where

$$\Delta = \{-\pi = x_n < x_{n-1} < ... < x_1 < x_0 < x_{n-1} < ... < x_n = \pi\}$$

such that $|x_i| = |x_i|, i = 1, n$, also contains only cosine terms.

Proof. Let $(S^3_{f\Delta})^* : [-\pi, \pi] \to \mathbb{R}, (S^3_{f\Delta})^* \in C^2[-\pi, \pi]$ be another function such that $S^3_{f\Delta}(-x) = (S^3_{f\Delta})^*(x)$ and $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$ where $a_k, k = 1, 2, ..$ are Fourier coefficients of function $f(x) \in C^2[-\pi, \pi]$.

Now we have

$$(S^3_{f\Delta})^*(x_i) = S^3_{f\Delta}(x_{i-}) = f(x_{i-}) = f(-x_i) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k(-x_i) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx_i = f(x_i), i = 1, n.$$
Consequently \((S^3_{f \Delta})^* : [-\pi, \pi] \to \mathbb{R}\) represents a spline for function \(f(x)\) and from the uniqueness of the interpolating spline we get that \((S^3_{f \Delta})^* = S^3_{f \Delta}\) respectively \(S^3_{f \Delta}(x-i) = S^3_{f \Delta}(-x_i) = S^3_{f \Delta}(x_i), i = \overline{1,n}\), which implies that the Fourier series of the spline contains only cosine terms. \(\square\)

**Theorem 3.3.** If the Fourier series of periodic function \(f(x) \in C^2[-\pi, \pi]\) contains only sine terms, then the Fourier series of the interpolating spline \(S^3_{f \Delta}\), where

\[
\Delta = \{-\pi = x_{-n} < x_{-(n-1)} < \ldots < x_{-1} < x_0 < x_1 < \ldots < x_{n-1} < x_n = \pi\}
\]

such that \(|x_{-i}| = |x_i|, i = \overline{1,n}\) also contains only sine terms.

**Proof.** Let \((S^3_{f \Delta})^* : [-\pi, \pi] \to \mathbb{R}, (S^3_{f \Delta})^* \in C^2[-\pi, \pi]\) be another function such that \((S^3_{f \Delta})^*(x) = -S^3_{f \Delta}(-x)\) and \(f(x) = \sum_{k=1}^{\infty} a_k \sin kx\) where \(a_k, k = 1, 2, \ldots\) are Fourier coefficients of function \(f(x) \in C^2[-\pi, \pi]\). Now we have

\[-(S^3_{f \Delta})^*(x_i) = S^3_{f \Delta}(x_{-i}) = f(x_{-i}) = f(-x_i) = \sum_{k=1}^{\infty} a_k \sin k(-x_i) =
\]

\[-\sum_{k=1}^{\infty} a_k \sin kx_i = -f(x_i), i = \overline{1,n}.
\]

Consequently \((S^3_{f \Delta})^* : [-\pi, \pi] \to \mathbb{R}\) represents a spline for function \(f(x)\) and from the uniqueness of the interpolating spline we get that \((S^3_{f \Delta})^* = S^3_{f \Delta}\) respectively \(S^3_{f \Delta}(x_{i-1}) = S^3_{f \Delta}(-x_i) = -S^3_{f \Delta}(x_i), i = \overline{1,n}\), which implies that the Fourier series of the spline contains only sine terms. \(\square\)

**Theorem 3.4.** Let \(f(x) \in C^2[0, 2\pi]\) and the spline \(s \in S^3_n\), be its interpolant, i.e. \(s(x_i) = f(x_i), i = \overline{1,n}\). If \(f\) and \(s\) satisfy the boundary conditions \(s'(0) = f'(0), s'(2\pi) = f'(2\pi), s''(0) = f''(0)\) and \(s''(2\pi) = f''(2\pi)\), then

\[
(||f - s||_2)^2 = (||f||_2 - ||s||_2)(||f||_2 + ||s||_2).
\]

**Proof.** We have that \((||f - s||_2)^2 = (||f||_2 - ||s||_2)(||f||_2 + ||s||_2) - 2\left(f - s, s\right)\).

Since \(f(x) \in C^2[0, 2\pi]\) and \(s \in C^2[a, b]\) has continuous derivatives of order 2, by successive integrations, using the boundary conditions and since \(s''(x) = 0\) we
find that
\[
\langle f - s, s \rangle_2 = - \int_0^{2\pi} (f(x) - s(x)) s''(x) \, dx \\
= - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (f(x) - s(x)) s''(x) \, dx \\
= - \sum_{i=1}^{n} \left. (f(x) - s(x)) s''(x) \right|_{x_{i-1}}^{x_i} = 0.
\]

Therefore
\[
(\|f - s\|_2)^2 = (\|f\|_2 - \|s\|_2)(\|f\|_2 + \|s\|_2).
\]

Theorem 3.5. Let \( f(x) = 0 \) and the spline \( s \in S^n_3 \) be its interpolant, i.e. \( s(x_i) = f(x_i), i = 1, n. \) If \( f \) and \( s \) satisfy the boundary conditions \( s'(0) = f'(0), \)
\( s'(2\pi) = f'(2\pi), s''(0) = f''(0) \) and \( s''(2\pi) = f''(2\pi), \) then \( s = 0. \)

Proof. For \( f(x) = 0, \) from Theorem 3.4, we have:
\[
(\|0 - s\|_2)^2 = 0 - (\|s\|_2)^2 \implies (\|s\|_2)^2 = 0 \implies \|s\|_2 = 0.
\]
Now from the boundary conditions since \( s^{(i)}(0) = s^{(i)}(2\pi) = 0, \) \( i = 1, 2 \implies s = 0. \)

References
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