ON WEAK FORMS OF PYTHAGOREAN NANO OPEN SETS

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ABSTRACT. The objective is to introduce the weakest form of Pythagorean nano open sets, namely Pythagorean nano semi-open, Pythagorean nano pre-open, Pythagorean nano $\alpha$-open and Pythagorean nano $\gamma$ and $\beta$ open sets. Various Pythagorean nano semi-open and Pythagorean nano $\alpha$-open corresponding to its different characterizations have also been derived.

1. INTRODUCTION

L. A. Zadeh introduced the concept of fuzzy set having elements with degree of membership [1]. Atanassov [2] established the idea of Intuitionistic fuzzy set comprising elements with membership and non-membership degree. Chang [3] introduced the idea of fuzzy topological space along with its base properties as open, closed and continuity were defined in 1968. Following this Lowen [4] gave another definition of fuzzy topological space. The concept of intuitionistic topological space with few fundamental properties was established by Coker [5].

Pythagorean fuzzy subset which is a special fuzzy subset was instituted by Yager [6, 7] which has applications in social and natural sciences. Even when intuitionistic fuzzy subsets cannot be used can be replaced by the Pythagorean fuzzy subset. The Pythagorean fuzzy topological space was introduced following the concept of Chang by Olgun [8].

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L. Thivagar [9] instituted the perception of nano topological spaces. The nano topology and intuitionistic fuzzy topology was combined to a new form named Intuitionistic nano topology in 2017 and was coined by Ramachandran and Stephan [10]. The concept of nano topology through neutrosophic sets [11] was also coined by L. Thivagar et. al. A new structure, namely Pythagorean nano topology which is a combo of nano topology and Pythagorean fuzzy topology was coined by D. Ajay and J. J. Charisma [12] along with few properties.

2. Preliminaries

A subset of a non-void set $X$ is said to be a fuzzy subset if it is as $A = \{ \langle a, \rho_A(a) \rangle : a \in X \}$ with $\rho_A : X \to [0,1]$ (membership degree). A subset $B = \{ \langle b, \sigma(b), \mu(b) \rangle : b \in X \}$ of $X$ with $\sigma : X \to [0,1], \mu : X \to [0,1]$ (membership and non-membership degree) is Intuitionistic fuzzy subset satisfying $\sigma(b) + \mu(b) \leq 1$. A non-empty subset $C$ of $X$ of the form $C = \{ \langle c, \sigma(c), \mu(c) \rangle : c \in X \}$ with $\sigma : X \to [0,1], \mu : X \to [0,1]$ as membership and non-membership degree satisfying $\sigma^2(c) + \mu^2(c) \leq 1$ is named as Pythagorean fuzzy subset.

The pair $(U,R)$ is said approximation space, where $U$ is a non-void set called as universe and $R$ an equivalence relation. Let $K \subseteq U$. The lower approximation of $K$ with respect to $R$ is denoted by $L_R(x), L_R(x) = \bigcup_{x \in U} R(x) : R(x) \subseteq K$. $R(x)$ is the equivalence relation determined by way of $x$. The upper approximation is denoted as $U_R(x)$ and described as $U_R(x) = \bigcup_{x \in U} R(x) : R(x) \cap K \neq \emptyset$. The boundary of $K$ is $B_R(x) = U_R(x) - L_R(x)$.

Let $U$ be an universe with equivalence relation $R$ and $\tau_R(K) = \{ \emptyset, U, L_R(x), B_R(x), U_R(x) \}$ in which $K \subseteq U$. $\tau_R(K)$ satisfies the axioms: $U, \emptyset \in \tau_R(K)$, union of elements of $\tau_R(K)$ is in $\tau_R(K)$, intersection of finite sub-collection of $\tau_R(K)$ is in $\tau_R(K)$. $(U, \tau_R(K))$ is termed the Nano topological space.

Let $A$ be a subset in a nano topological space $(U, \tau_R(X))$. It is named as nano semi open, nano pre-open, nano $\alpha$-open if $A \subseteq Ncl(Nint(A)), A \subseteq Nint(Ncl(Nint(A)))$ respectively [13].

A non-empty set $J$ with the collection of Pythagorean fuzzy subsets $\rho$ is called Pythagorean fuzzy topological space if the following axioms are satisfied $1_X, 0_X \in \rho$, for any $S_1, S_2 \in \rho, S_1 \cap S_2 \in \rho$, for any $\{S_i\}_{i \in I} \subseteq \rho$ for all $i$, $\cup S_i \in \rho$ where $I$ is an arbitrary index set. The pair $(V, \mathfrak{R})$ is said to be a Pythagorean approximation.
space with \( D \) a Pythagorean set in \( V \) having \( \theta_D, \omega_D \) as membership and non-membership degree. The Pythagorean nano lower, Pythagorean nano upper and boundary of \( D \) are:

\[
PNL_{\text{R}}(D) = \{ \langle x, \theta_{LD}(x), \omega_{LD}(x) | z \in [x]_{\text{R}}, x \in V \rangle \}
\]

\[
PNU_{\text{R}}(D) = \{ \langle x, \theta_{RD}(x), \omega_{RD}(x) | z \in [x]_{\text{R}}, x \in V \rangle \}
\]

\[
PNB_{\text{R}}(D) = PNU_{\text{R}}(D) - PNL_{\text{R}}(D)
\]

where

\[
\theta_{LD}(x) = \bigwedge_{y \in [x]_{\text{R}}} \theta_D(y)
\]

\[
\omega_{LD}(x) = \bigvee_{y \in [x]_{\text{R}}} \omega_D(y)
\]

and

\[
\theta_{RD}(x) = \bigvee_{y \in [x]_{\text{R}}} \theta_D(y)
\]

\[
\omega_{RD}(x) = \bigwedge_{y \in [x]_{\text{R}}} \omega_D(y)
\]

respectively [12].

The pair \((V, \mathcal{R})\) along with \( \tau_{\text{R}}(Y) = \{ \emptyset_P, V_P, PNL_{\text{R}}(Y), PNU_{\text{R}}(Y), PNB_{\text{R}}(Y) \} \) is said to be Pythagorean nano topological space while satisfying these conditions: \( \emptyset_P, V_P \in \tau_{\text{R}}(Y) \), if \( A_i \in \tau_{\text{R}}(Y) \) for \( i = 1, 2, \ldots \) then \( \bigcup_{i=1}^{\infty} A_i \in \tau_{\text{R}}(Y) \), if \( A_i \in \tau_{\text{R}}(Y) \) for \( i = 1, 2, \ldots \) then \( \bigcap_{i=1}^{n} A_i \in \tau_{\text{R}}(Y) \) where \( \emptyset_P = \{ \langle x, 0, 1 | \forall x \in V \rangle \}, V_P = \{ \langle x, 1, 0 | \forall x \in V \rangle \} \) [12].

3. **Pythagorean nano \( \alpha \) open sets**

**Definition 3.1.** Let \((\mathcal{U}, \tau_{\text{R}}(X))\) be a PNT space and \( A \subseteq \mathcal{U} \).

1. If \( A \subseteq P\mathcal{N} \text{cl} (P\mathcal{N} \text{int}(A)) \), then \( A \) is Pythagorean nano semi-open (PNSO)
2. If \( A \subseteq P\mathcal{N} \text{int} (P\mathcal{N} \text{cl}(A)) \), then \( A \) is Pythagorean nano pre-open if (PNPO)
3. If \( A \subseteq P\mathcal{N} \text{int} (P\mathcal{N} \text{cl} (P\mathcal{N} \text{int}(A))) \), then \( A \) is Pythagorean nano \( \alpha \)-open (PN\(\alpha\)O).
Let PNSO($\mathcal{U}, X$), PNPO($\mathcal{U}, X$) and $\tau^\alpha_{\text{Pr}}(X)$ be the families of all Pythagorean nano semi-open, Pythagorean nano pre-open and Pythagorean nano $\alpha$ open subsets of $\mathcal{U}$ respectively. Let $A \subseteq \mathcal{U}$. $A$ is said to be a Pythagorean nano semi-closed, Pythagorean nano pre-closed, Pythagorean nano $\alpha$ closed if its complement is PNSO, PNPO, PN $\alpha$O respectively.

**Example 1.** Let $\mathcal{U}=\{a, b, c\}$ be the Universe and let $\mathcal{U}/R = \{\{a\}, \{c\}\}$. Let us consider

$$A = \{\langle a, 0.7, 0.6 \rangle, \langle b, 0.5, 0.4 \rangle, \langle c, 0.6, 0.3 \rangle\}$$

Then

$$\text{PL}_R(A) = \{\langle a, 0.5, 0.6 \rangle, \langle b, 0.5, 0.6 \rangle, \langle c, 0.6, 0.3 \rangle\}$$

$$\text{PU}_R(A) = \{\langle a, 0.7, 0.4 \rangle, \langle b, 0.7, 0.4 \rangle, \langle c, 0.6, 0.3 \rangle\}$$

$$\text{PB}_R(A) = \{\langle a, 0.5, 0.4 \rangle, \langle b, 0.5, 0.4 \rangle, \langle c, 0.4, 0.7 \rangle\}$$

$$\tau_R(A) = \left\{\emptyset, \mathcal{U}_p, \{\langle a, 0.5, 0.6 \rangle, \langle b, 0.5, 0.6 \rangle, \langle c, 0.6, 0.3 \rangle\}, \{\langle a, 0.7, 0.4 \rangle, \langle b, 0.7, 0.4 \rangle, \langle c, 0.6, 0.3 \rangle\}, \{\langle a, 0.5, 0.4 \rangle, \langle b, 0.5, 0.4 \rangle, \langle c, 0.4, 0.7 \rangle\}\right\}$$

The Pythagorean nano closed sets are $\emptyset$, $\mathcal{U}_p$, $\{\langle a, 0.5, 0.4 \rangle, \langle b, 0.5, 0.4 \rangle, \langle c, 0.4, 0.7 \rangle\}$, $\{\langle a, 0.3, 0.6 \rangle, \langle b, 0.3, 0.6 \rangle, \langle c, 0.4, 0.7 \rangle\}$, $\{\langle a, 0.5, 0.6 \rangle, \langle b, 0.5, 0.6 \rangle, \langle c, 0.6, 0.3 \rangle\}$. In this example, the set $A$ is PNSO, PNPO, PN $\alpha$O because it satisfies the required conditions $A \subseteq \mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A))$, $A \subseteq \mathcal{P}N_{\text{int}}(\mathcal{P}N_{\text{cl}}(A))$, $A \subseteq \mathcal{P}N_{\text{int}}(\mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A)))$ respectively.

**Theorem 3.1.** If $A$ is PNO in $(\mathcal{U}, \tau_R(X))$, then it is a PN $\alpha$O in $\mathcal{U}$.

**Proof.** Since $A$ is PNO in $\mathcal{U}$, $\mathcal{P}N_{\text{int}}(A) = A$. Now:

$$\mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A)) = \mathcal{P}N_{\text{cl}}(A) \supseteq A$$

$$\Rightarrow A \subseteq \mathcal{P}N_{\text{cl}}(A) = \mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A))$$

$$\Rightarrow \mathcal{P}N_{\text{int}}(A) \subseteq \mathcal{P}N_{\text{int}}(\mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A)))$$

$$\Rightarrow A \subseteq \mathcal{P}N_{\text{int}}(\mathcal{P}N_{\text{cl}}(\mathcal{P}N_{\text{int}}(A)))$$

Therefore, $A$ is PN $\alpha$O in $\mathcal{U}$. \hfill \square

**Theorem 3.2.** $\tau^\alpha_{\text{Pr}}(X) \subseteq \text{PNSO} (\mathcal{U}, X)$ in a PNT space $(\mathcal{U}, \tau_R(X))$. 
Proof. If $A \in \tau_{PR}^{\alpha}(X)$, then $A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N} \mathcal{E} \mathcal{L} \mathcal{P}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N}(A))$. Consider any set $D$ with interior $\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(D) \subseteq D$. Therefore

$$\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A)) \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$$

$$\Rightarrow A \in \mathrm{PNSO}(\mathcal{U}, X)$$

Thus $\tau_{PR}^{\alpha}(X) \subseteq \mathrm{PNSO}(\mathcal{U}, X)$. □

Theorem 3.3. $\tau_{PR}^{\alpha}(X) \subseteq \mathrm{PNPO}(\mathcal{U}, X)$ in a PNT space $(\mathcal{U}, \tau_{R}(X))$.

Proof. If $A \in \tau_{PR}^{\alpha}(X)$, then $A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N} \mathcal{E} \mathcal{L} \mathcal{P}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N}(A))$. Since $\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \subseteq A$, $\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A)), \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A)) \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A)) \Rightarrow A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$. Thus $A \in \mathrm{PNPO}(\mathcal{U}, X)$. □

Theorem 3.4. $\tau_{PR}^{\alpha}(X) = \mathrm{PNPO}(\mathcal{U}, X) \cap \mathrm{PNSO}(\mathcal{U}, X)$ in a PNT space $(\mathcal{U}, \tau_{R}(X))$.

Proof. Let $A \in \tau_{PR}^{\alpha}(X)$, then $A \in \mathrm{PNSO}(\mathcal{U}, X)$ and also $A \in \mathrm{PNPO}(\mathcal{U}, X)$ (by the previous theorems). Thus

$$A \in \mathrm{PNPO}(\mathcal{U}, X) \cap \mathrm{PNSO}(\mathcal{U}, X) \Rightarrow \tau_{PR}^{\alpha}(X) \subseteq \mathrm{PNPO}(\mathcal{U}, X) \cap \mathrm{PNSO}(\mathcal{U}, X)$$

Conversely, if $A \in \mathrm{PNPO}(\mathcal{U}, X) \cap \mathrm{PNSO}(\mathcal{U}, X)$, then:

(3.1) $A \in \mathrm{PNSO}(\mathcal{U}, X) \Rightarrow A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N} \mathcal{E} \mathcal{L} \mathcal{P}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O} \mathcal{N}(A))$

(3.2) $A \in \mathrm{PNPO}(\mathcal{U}, X) \Rightarrow A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$

From (3.1),

$$\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A) \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$$

$$\Rightarrow A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$$

Thus

(3.3) $\mathrm{PNSO}(\mathcal{U}, X) \subseteq \tau_{PR}^{\alpha}(X)$

From (3.2),

$$A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A)) \Rightarrow A \subseteq \mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(\mathcal{P} \mathcal{N} \mathcal{I} \mathcal{R} \mathcal{N} \mathcal{I} \mathcal{O}(A))$$

Thus

(3.4) $\mathrm{PNPO}(\mathcal{U}, X) \subseteq \tau_{PR}^{\alpha}(X)$

Equations (3.3) and (3.4) imply that $\mathrm{PNSO}(\mathcal{U}, X) \cap \mathrm{PNPO}(\mathcal{U}, X) \subseteq \tau_{PR}^{\alpha}(X)$.

Therefore $\tau_{PR}^{\alpha}(X) = \mathrm{PNPO}(\mathcal{U}, X) \cap \mathrm{PNSO}(\mathcal{U}, X)$. □
Theorem 3.5. If in a PNT space \((U, \tau_R(X))\), PL\(_R(X) = PU\_R(X) = X\), then \(U_p, \emptyset_p\), PNL\(_R(X) (= PNU\_R(X))\) and any set \(A\) such that PNL\(_R(X) \subseteq A\) is the only PN \(\alpha\)O set in \(U\).

Proof. Since PNL\(_R(X) = PNU\_R(X) = X\), the PNT is \(\tau_R(X) = \{U_p, \emptyset_p, PNL\_R(X)\}\). From Theorem 3.3, any PN open set is PN \(\alpha\)O set. Thus \(U_p, \emptyset_p, PNL\_R(X)\) are all PN \(\alpha\)O. Let us assume \(A \subset PNL\_R(X)\), then \(PN\text{int}(A) = \emptyset_p\), because the only open set contained in \(A\) is \(\emptyset_p\). Then \(PN\overline{cl}(PN\text{int}(A)) = PN\overline{cl}(\emptyset_p) = \emptyset_p\). Thus \(A\) is not a PN \(\alpha\)O set. If \(PNU\_R(X) \subset A\), then \(PNU\_R(X)\) is the largest PN open subset of \(A\) (i.e. \(PN\text{int}(A) = PNL\_R(X)\)).

\[
PN\text{int}(PN\overline{cl}(PN\text{int}(A))) = PN\text{int}(PN\overline{cl}(PL\_R(X))) \Rightarrow PN\text{int}(U_p) = U_p
\]
Thus \(A \subseteq PN\text{int}(PN\overline{cl}(PN\text{int}(A))) \Rightarrow A\) is a PN \(\alpha\)O.

Theorem 3.6. \(U_p, \emptyset_p, PNU\_R(X)\) and any set that contains PNU\(_R(X)\) are the only PN \(\alpha\)O sets in a PNT space if PNL\(_R(X) = \emptyset_p\).

Proof. Since PNL\(_R(X) = \emptyset_p\), PNB\(_R(X) = PNU\_R(X)\). Thus the PNT is \(\{U_p, \emptyset_p, PU\_R(X)\}\) and by Theorem 3.3 the members of PNT are all PN \(\alpha\)O sets. Let \(A \subset PNU\_R(X)\). If \(A \subset PU\_R(X)\), then \(A\) is not a PN \(\alpha\)O set. If \(PNU\_R(X) \subset A\), then \(PNU\_R(X)\) is the largest PN open subset of \(A\) (i.e. \(PN\text{int}(A) = PNL\_R(X)\)).

\[
PN\text{int}(PN\overline{cl}(PN\text{int}(A))) = PN\text{int}(PN\overline{cl}(PL\_R(X))) \Rightarrow PN\text{int}(U_p) = U_p
\]
Thus \(A \subseteq PN\text{int}(PN\overline{cl}(PN\text{int}(A))) \Rightarrow A\) is a PN \(\alpha\)O.

Theorem 3.7. If PNU\(_R(X) = U_p\) and PNL\(_R(X)\) is non-empty in a PNT space \((U, \tau_R(X))\), then \(U_p, \emptyset_p, PNL\_R(X) \& PNU\_R(X)\) are the only PN \(\alpha\)O sets in \(U\).

Proof. Since PNU\(_R(X) = U_p\) and PNU\(_R(X) = \emptyset_p\), the PNO sets in \(U\) are \(U_p, \emptyset_p, PNL\_R(X)\) and \(PNU\_R(X)\) and hence they are also PN \(\alpha\)O. If \(A = \emptyset\), then obviously it is PN \(\alpha\)O.

Now, let \(A\) be non-empty, and let \(A \subset PNL\_R(X)\). It implies \(PN\text{int}(A) = \emptyset_p\), because \(\emptyset_p\) is the largest PNO in \(A\) and hence \(A \not\subset PN\text{int}(PN\overline{cl}(PN\text{int}(A)))\). This implies \(A\) is not PN \(\alpha\)O set. When PNL\(_R(X) \subset A\), \(PN\text{int}(A) = PNL\_R(X)\) and therefore

\[
PN\text{int}(PN\overline{cl}(PN\text{int}(A))) = PN\text{int}(PN\overline{cl}(PNU\_R(X))) = PL\_R(X) \subset A
\]
\[ \Rightarrow A \not\subseteq \mathcal{P}N \text{int} (\mathcal{P}N c\ell (\mathcal{P}N \text{int} (A))) \]

Therefore A is not PN $O$ set.

Similarly it can be proved that A is not PN $O$ for $P_{NB}$ $R$ $X$ $A$ and $A \subseteq P_{NB}$ $R$ $X$. If A has one element in $P_{NB}$ $R$ $X$ or $P_{NL}$ $R$ $X$, then A is not PN $O$ set. $U_p, \emptyset_p, P_{NL}$ $R$ $X$ & $P_{NU}$ $R$ $X$ are the only PN $O$ sets in $U$ when $P_{NU}$ $R$ $X$ $=U_p$ and $P_{NL}$ $R$ $X$ $=\emptyset_p$. □

**Corollary 3.1.** $\tau^0_{PL}$ $(X) = \tau_R (X)$, if $P_{NU}$ $R$ $(X) = U_p$.

**Theorem 3.8.** Let $P_{NL}$ $R$ $X$ $\neq P_{NU}$ $R$ $X$ where $P_{NL}$ $R$ $X$ $\neq \emptyset_p$ and $P_{NU}$ $R$ $X$ $\neq U_p$ in a PNT space $(U, \tau_R (X))$. Then $U_p, \emptyset_p, P_{NL}$ $R$ $X$, $P_{NU}$ $R$ $X$, $P_{NB}$ $R$ $X$ and any set $A$ such that it contains $P_{NU}$ $R$ $X$ are the only PN $O$ sets in $U$.

**Proof.** The PNT on $U$ is given by $\tau_R (X) = \{U_p, \emptyset_p, P_{NL}$ $R$ $X, P_{NU}$ $R$ $X, P_{NB}$ $R$ $X\}$ and hence $U_p, \emptyset_p, P_{NL}$ $R$ $X, P_{NU}$ $R$ $X, P_{NB}$ $R$ $X$ are PN $O$ sets in $U$. Let A be any PN set in $U$ such that $P_{NU}$ $R$ $X$ $\subseteq A$, then $\mathcal{P}N \text{int} (A) = P_{NU}$ $R$ $X$. Therefore $\mathcal{P}N \text{int} (\mathcal{P}N c\ell (\mathcal{P}N \text{int} (A))) = \mathcal{P}N \text{int} (\mathcal{P}N c\ell (P_{NU}$ $R$ $X)) = \mathcal{P}N \text{int} (U_p)$.

Thus $A \subseteq \mathcal{P}N \text{int} (\mathcal{P}N c\ell (\mathcal{P}N \text{int} (A))) \iff A$ is PN $O$ in $U$ when $P_{NU}$ $R$ $X$ $\subseteq A$. When $A \subseteq P_{NL}$ $R$ $X$, $\mathcal{P}N \text{int} (A) = \emptyset_p \iff \mathcal{P}N \text{int} (\mathcal{P}N c\ell (\mathcal{P}N \text{int} (A))) = \emptyset_p$. Thus $A$ is not a PN $O$ set. When $A \subseteq P_{NU}$ $R$ $X$ but is neither a subset of $P_{NL}$ $R$ $X$ nor of $P_{NB}$ $R$ $X$, then $\mathcal{P}N \text{int} (A) = \emptyset_p \iff A$ is not a PN $O$ set. Thus $U_p, \emptyset_p, P_{NL}$ $R$ $X, P_{NU}$ $R$ $X, P_{NB}$ $R$ $X$ and any subset $P_{NU}$ $R$ $X$ $\subseteq A$ are the only PN$O$ sets. □

**4. Forms of Pythagorean nano semi open and Pythagorean nano regular open sets**

**Remark 4.1.** $U_p, \emptyset_p$ are always Pythagorean Nano Semi Open(PNSO) since $\mathcal{P}N c\ell (\mathcal{P}N \text{int} (U_p)) = U_p$ and $\mathcal{P}N c\ell (\mathcal{P}N \text{int} (\emptyset_p)) = \emptyset_p$.

**Theorem 4.1.** Let $(U, \tau_R (X))$ be a PNT space with $P_{NU}$ $R$ $X = P_{NL}$ $R$ $X$, then $\emptyset_p$ and sets $A$ such that $P_{NL}$ $R$ $X \subseteq A$ are the only PNSO subsets of $U$.

**Proof.** Since $P_{NU}$ $R$ $X = P_{NL}$ $R$ $X$, $\tau_R (X) = \{U_p, \emptyset_p, P_{NL}$ $R$ $X\}$. $\mathcal{P}N \text{int} (\emptyset_p) = \emptyset_p$, $\mathcal{P}N c\ell (\mathcal{P}N \text{int} (\emptyset_p)) = \emptyset_p$ implying $\emptyset_p$ is PNSO. Let $A \neq \emptyset_p$ be a subset of $U$ and $A \subseteq P_{NL}$ $R$ $X$, then $\mathcal{P}N c\ell (\mathcal{P}N \text{int} (A)) = \mathcal{P}N c\ell (\emptyset_p) = \emptyset_p$. If $A \subseteq P_{NL}$ $R$ $X$ then A is not a PNSO.
Consider $\mathcal{P}_R(X) \subseteq A$, thus $\mathcal{P}_R(\mathcal{P} \text{int}(A)) = \mathcal{P}_R(\mathcal{P}_R(X)) = \mathcal{U}_p$ (since $\mathcal{P}_R(X) = \mathcal{P}_R(X)$). Thus $A \subseteq \mathcal{P}_R(\mathcal{P} \text{int}(A))$ and $A$ is PNSO. Therefore $\emptyset_p$ and any set which contains $\mathcal{P}_R(X)$ are the only PNSO sets in $\mathcal{U}$ whenever $\mathcal{P}_R(X) = \emptyset_p$. Thus $A \subseteq \mathcal{P}_R(\mathcal{P} \text{int}(A))$ and $A$ is PNSO. Therefore $\emptyset_p$ and any set which contains $\mathcal{P}_R(X)$ are the only PNSO sets in $\mathcal{U}$ whenever $\mathcal{P}_R(X) = \emptyset_p$. □

**Theorem 4.2.** If $\mathcal{P}_R(X) = \emptyset_p$ and $\mathcal{P}_R(X) \neq \mathcal{U}_p$, then only those sets containing $\mathcal{P}_R(X)$ are the PNSO sets in $\mathcal{U}$.

**Proof.** Let $\tau_R(X) = \{\mathcal{U}_p, \emptyset_p, \mathcal{P}_R(X)\}$ and $A$ be a non-empty subset of $\mathcal{U}$. If $A \subseteq \mathcal{P}_R(X)$, then $\mathcal{P} \text{int}(A) = \emptyset_p \implies \mathcal{P}_R(\mathcal{P} \text{int}(A)) = \emptyset_p$, thus $A$ is not PNSO.

Consider $\mathcal{P}_R(X) \subseteq A$, then $\mathcal{P} \text{int}(A) = \mathcal{P}_R(X) \implies \mathcal{P}_R(\mathcal{P} \text{int}(A)) = \mathcal{U}_p$. This implies that $A$ is a PNSO set. Thus the sets $A$, a superset of $\mathcal{P}_R(X)$ are the only PNSO sets in $\mathcal{U}$ whenever $\mathcal{P}_R(X) = \emptyset_p$ and $\mathcal{P}_R(X) \neq \mathcal{U}_p$. □

**Theorem 4.3.** If $\mathcal{P}_R(X) = \mathcal{U}_p$ is a PNT space, then $\mathcal{U}_p, \emptyset_p, \mathcal{P}_R(X)$ and $\mathcal{P}_R(X)$ are the only PNSO sets in $\mathcal{U}$.

**Proof.** Let $\tau_R(X) = \{\mathcal{U}_p, \emptyset_p, \mathcal{P}_R(X), \mathcal{P}_B(X)\}$ and $A$ be a non-empty subset of $\mathcal{U}$. Obviously $A$ is not PNSO set when $A \subseteq \mathcal{P}_R(X)$. If $A = \mathcal{P}_R(X)$ then $\mathcal{P}_R(\mathcal{P}_R(X)) = \mathcal{P}_R(X)$ and hence $A \subseteq \mathcal{P}_R(\mathcal{P}_R(X))$ which implies that $A$ is a PNSO set. But $A$ is not a PNSO set when $\mathcal{P}_R(X) \subseteq A$ since $\mathcal{P}_R(\mathcal{P}_R(X)) = \mathcal{P}_R(X)$ but $A \not\subseteq \mathcal{P}_R(X)$.

Similarly if $A \subseteq \mathcal{P}_B(X)$ and $\mathcal{P}_R(X) \subseteq A$, then $\mathcal{P}_R(\mathcal{P}_B(X)) = \emptyset_p$ and $\mathcal{P}_R(\mathcal{P}_B(X)) = \mathcal{P}_B(X)$ respectively. Thus $A \not\subseteq \mathcal{P}_R(\mathcal{P}_B(X))$. Therefore $A$ is not a PNSO set. If $A$ has at least one element in $\mathcal{P}_R(X)$ and at least one element in $\mathcal{P}_B(X)$, then $A$ is not a PNSO. Thus $\mathcal{U}_p, \emptyset_p, \mathcal{P}_R(X), \mathcal{P}_B(X)$ are the only PNSO sets in $\mathcal{U}$ when $\mathcal{P}_R(X) = \mathcal{U}_p$ and $\mathcal{P}_R(X) \neq \emptyset_p$.

**Corollary 4.1.** If $\mathcal{P}_R(X) = \emptyset_p$ in Theorem 4.3, then $\emptyset_p$ and $\mathcal{U}_p$ are the only PNSO sets in $\mathcal{U}$.

**Theorem 4.4.** If $A$ and $B$ are PNSO in $\mathcal{U}$, then the union is also a PNSO in $\mathcal{U}$.

**Proof.** Since $A$ and $B$ are PNSO, $A \subseteq \mathcal{P}_R(\mathcal{P} \text{int}(A))$ and $B \subseteq \mathcal{P}_R(\mathcal{P} \text{int}(B))$.

Consider $A \cup B$.

$$A \cup B \subseteq \mathcal{P}_R(\mathcal{P} \text{int}(A)) \cup \mathcal{P}_R(\mathcal{P} \text{int}(B))$$
\[ \subseteq \mathcal{PN}_{\text{cl}}(\mathcal{PN}_{\text{int}}(A) \cup \mathcal{PN}_{\text{int}}(B)) \]

since \( \mathcal{PN}_{\text{int}}(A) \cup \mathcal{PN}_{\text{int}}(B) = \mathcal{PN}_{\text{int}}(A \cup B) \). Therefore the union of two PNSO sets is a PNSO set, but intersection of two PNSO sets need not be PNSO.

\[ \square \]

**Definition 4.1.** A subset \( A \) of a PNT space \( (U, \tau_R(X)) \) is Pythagorean Nano Regular Open (PNRO) in \( U \), if \( \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = A \).

**Theorem 4.5.** Any PNRO set is PNO.

**Proof.** If \( A \) is PNRO in \( U \), then \( \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = A \). Since

\[ \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = A \Rightarrow \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = A \]

Thus \( \mathcal{PN}_{\text{int}}(A) = A \). Therefore \( A \) is a PNO set in \( U \). But the converse of the theorem need not be true.

\[ \square \]

**Theorem 4.6.** In a PNT space \( (U, \tau_R(X)) \), if \( \mathcal{PNL}_R(X) = \mathcal{PNU}_R(X) \), then the only PNRO sets are \( U_p \) and \( \emptyset_p \).

**Proof.** The PNO sets in \( U \) are \( U_p, \emptyset_p \) and

\[ \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(\mathcal{PNL}_R(X))) = U_p \neq \mathcal{PNL}_R(X) \]

We have \( \mathcal{PNL}_R(X) \subseteq U_p \) but for PNO \( \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) = A \). Thus \( \mathcal{PNL}_R(X) \) is not PNRO. Therefore the only PNRO sets are \( U_p \) and \( \emptyset_p \).

\[ \square \]

**Definition 4.2.** Let \( (U, \tau_R(X)) \) be a PNT space and \( A \subseteq U \). Then \( A \) is Pythagorean nano \( \gamma \) open \( (\mathcal{PN}_{\gamma O}) \) and Pythagorean nano \( \beta \) open \( (\mathcal{PN}_{\beta O}) \) if

\[ A \subseteq \mathcal{PN}_{\text{cl}}(\mathcal{PN}_{\text{int}}(A)) \cup A \subseteq \mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A)) \]

and

\[ A \subseteq \mathcal{PN}_{\text{cl}}(\mathcal{PN}_{\text{int}}(\mathcal{PN}_{\text{cl}}(A))) \]

respectively.

**Definition 4.3.** Let \( (U, \tau_R(X)) \) and \( (V, \tau_R(Y)) \) be PNT spaces. The mapping \( f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is said to be \( \mathcal{PN}_\beta \) continuous if \( f^{-1}(A) \) is \( \mathcal{PN}_\beta \) in \( U \) for every PNO \( A \) in \( V \).
Proposition 4.1. For every PNT space \((U, \tau_R(X))\), we have that: \(\text{PNSO}(U, X) \cup \text{PNPO}(U, X) \subseteq (P\text{N}^\gamma\text{O}) \subseteq (P\text{N}^\beta\text{O})\) holds but none of them are vice versa.

Proposition 4.2. Let \((U, \tau_R(X))\) be a PNT space then:

1. If \(A \subseteq U\) is PNO and \(B \subseteq U\) is PNSO (resp. PNPO, P\text{N}^\beta\text{O}, P\text{N}^\gamma\text{O}) then \(A \cap B\) is PNSO (resp. PNPO, P\text{N}^\beta\text{O}, P\text{N}^\gamma\text{O}).

2. For every subset \(A \subseteq U\), \(A \cap P\text{N}int(P\text{N}cl(A))\) is PNPO.

3. \(A \subseteq U\) is P\text{N}^\gamma\text{O}, if and only if \(A\) is union of PNSO and PNPO.

5. Conclusion

Herein, weak Pythagorean nano open sets have been introduced. Along with it, various forms of Pythagorean nano semi-open, Pythagorean nano pre-open and Pythagorean nano \(\alpha\)-open sets were derived. Furthermore, strongest forms of Pythagorean nano open sets have been studied.

References


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