TESTING BHNBU CLASS OF LIFE-TIME DISTRIBUTION BASED ON MOMENT INEQUALITY

A. TOUSEEF AHMED AND U. RIZWAN

ABSTRACT. In this paper, a new test statistic, for testing BHNBUE class of life-time distribution based on the moment inequality is proposed. This inequality demonstrates that, if the mean life is finite, then all higher order moments exist. Using Monte Carlo method, critical values of the proposed test is calculated for \( n = 6(1)40 \) and tabulated. Finally, application to real-life data are carried out.

1. INTRODUCTION

Ageing Classes of life-time distributions are defined to categorize the life-time distributions according to their ageing properties. The main aim of constructing new tests is to gain higher efficiencies. Testing bivariate exponentiality against some classes of life-time distributions have been introduced by various researchers from different point of views. Testing bivariate exponentiality against BHNBUE ageing class of life-time distribution has seen a good deal of attention. For testing bivariate exponentiality against BHNBUE alternative can be found in the work of Kanwar Sen and Madhu Bala Jain in [2]. Now we propose a test statistic testing BHNBUE class against bivariate exponential distribution, based on the moment inequality.

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The rest of paper is arranged as follows; In section 2, the preliminaries required for further discussion is given. In section 3, moment inequality for the Bivariate Harmonic New Better than Used in Expectation (BHNUE) class of life-time distribution is derived. A new test statistic for testing BHNUE class against bivariate exponential distribution, based on the moment inequality is proposed in section 4. Using Monte Carlo Method critical values of the proposed test statistic are calculated for \( n = 6(1)40 \) and tabulated in section 5. The application of the proposed test to real data sets is discussed in section 6. Finally, conclusion is given in section 7.

2. Definitions

Let \((X, Y)\) denote the survival time of a device having a joint distribution function \(F(x, y)\). The bivariate joint survival function is given by

\[
\bar{F}(x, y) = P(X > x, Y > y), \quad x, y \geq 0,
\]

where it is assumed that \(\bar{F}(0,0) = 1\).

The following definitions of Bivariate ageing classes of life-time distributions appeared in [5].

**Definition 2.1.** A bivariate random variable \((X, Y)\) or its distribution \(\bar{F}(t, s)\) is said to have Bivariate Harmonic New Better than Used in Expectation (BHNUE) ageing class, if

\[
\int_{0}^{x} \int_{0}^{y} \bar{F}(u, v)dvdu \leq \mu \exp \left[ -\frac{x + y}{\mu} \right],
\]

for all \(x, y, t, s \geq 0\), where \(\mu = \int_{0}^{\infty} \int_{0}^{\infty} \bar{F}(x, y)dydx\) denotes the mean of \(\bar{F}\) and is assumed to be finite.

**Definition 2.2.** The \(r^{th}\) Moment of a bivariate random variable \((X, Y)\) is

\[
\mu_{(r)} = E(X^{r}Y^{r}) = r^{2} \int_{0}^{\infty} \int_{0}^{\infty} (xy)^{r-1} \bar{F}(x, y)dydx.
\]

3. Moment Inequality

In this section, the moment inequality for BHNUE class is derived.
Theorem 3.1. Let \( F \) be a BHNBUE ageing class of life-time distribution such that moment of all orders exist and is finite. Then \( \frac{\mu_{r+2}}{(r+2)!} \leq \mu^{r+2}(r!) \) for \( r \geq 0 \).

Proof. Let \( G(x,y) = \int_0^x \int_0^y F(u,v)dvdu. \) Since \( F \) is BHNBUE, we have

\[
G(x,y) \leq \mu \exp \left[ -\frac{x+y}{\mu} \right].
\]

Multiplying both sides by \( x^r y^r \), for \( r \geq 0 \) and integrating twice over \([0, +\infty)\) with respect to \( x \) and \( y \), we get

\[
\int_0^\infty \int_0^\infty x^r y^r G(x,y)dydx \leq \mu \int_0^\infty \int_0^\infty x^r y^r \exp \left[ -\frac{x+y}{\mu} \right] dydx
\]

\[
= \mu \int_0^\infty x^r \exp \left( -\frac{x}{\mu} \right) dx \int_0^\infty y^r \exp \left( -\frac{y}{\mu} \right) dy
\]

\[
= \mu^{r+2}(r!)^2.
\]

Therefore \( \frac{\mu_{r+2}}{(r+1)(r+2)} \leq \mu^{r+2}(r!) \) (or) \( \frac{\mu_{r+2}}{(r+2)!} \leq \mu^{r+2}(r!) \).

This completes the proof of theorem. \( \square \)
Remark 3.1. For \( r = 0 \), the above inequality reduces to \( \frac{\mu(2)}{2!} \leq \mu^2 \), (or) \( \mu(2) \leq 2\mu^2 \).

4. TESTING AGAINST BHNBUE ALTERNATIVES

Using the above inequality, we test the null hypothesis

\( H_0 : F \) is bivariate exponential against

\( H_1 : F \) is BHNBUE and not bivariate exponential.

Consider the bivariate exponential distribution introduced by Marshall and Olkin in [4], given by

\[
F(x, y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)), \quad \text{for all} \quad x, y, \lambda_1, \lambda_2 > 0 \quad \text{and} \quad \lambda_{12} \geq 0,
\]

where

\[
\lambda_1 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_2}, \quad \lambda_2 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_1}, \quad \lambda_{12} = \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \frac{\mu_{12} - \mu_1 \mu_2}{\mu_{12}}.
\]

\[
\mu_1 = \int_0^\infty F(x, 0) dx, \quad \mu_2 = \int_0^\infty F(0, y) dy \quad \text{and} \quad \mu_{12} = \int_0^\infty \int_0^\infty F(x, y) dx dy.
\]

Define \( \delta_\text{E} = 2\mu^2 - \mu(2) \geq 0 \).

Note that under \( H_0 \), \( \delta_\text{E} = 0 \), while under \( H_1 \), \( \delta_\text{E} > 0 \). Let \( (X_1, X_2), \ldots, (X_{i-1}, X_i), \ldots, (X_n, X_{n+1}) \) be a random sample from a distribution \( F \). The empirical estimate \( \hat{\delta}_\text{E} \) of \( \delta_\text{E} \) can be obtained as

\[
\hat{\delta}_\text{E} = \frac{1}{n^2} \sum_i \sum_j \{2X_i^2 - X_j^2\} = \frac{1}{n^2} \sum_i \sum_j \phi(X_i, X_j)
\]

where \( \phi(X_i, X_j) = 2X_i^2 - X_j^2 \), to make the test Statistic scale invariant, let

\[
\hat{\Delta} = \frac{\hat{\delta}_\text{E}}{X^2}.
\]

Define the symmetric kernel \( \eta(X_i, X_j) = \frac{1}{n^2} \sum \phi(X_i, X_j) \), where the sum is taken over all arrangement of \( X_i, X_j \). Then \( \hat{\Delta} \) is equivalent to the classical U-statistic, [3] and is given by \( U_n = \frac{1}{\binom{n}{2}} \sum_{i < j} \eta(X_i, X_j) \). The asymptotic normality of \( \hat{\Delta} \) is summarized in the following theorem.

Theorem 4.1. As \( n \to \infty \), \( \sqrt{n} (\hat{\Delta} - \delta_\text{E}) \) is asymptotically normal with mean 0 and variance \( \sigma^2 \), where \( \sigma^2 = \text{Var}\{2X^2 - 2\mu_2 - X^2 - \mu_2\} \). Under \( H_0 \), the variance reduces to \( \sigma^2 = \text{Var}\{X^2 - 2\} = 20 \).

Proof. From the standard theory of U-Statistics, [3], \( \sigma^2 = \text{Var}\{\varsigma(X_i, X_j)\} \), where \( \varsigma(X_i, X_j) = E[\phi(X_1, X_2) \mid X_1] + E[\phi(X_1, X_2) \mid X_2] \).
$E[\phi(X_1, X_2) | X_1] = 2X_1^2 - \mu_2$, $E[\phi(X_1, X_2) | X_2] = 2\mu_2 - X_2^2$ and $\varsigma(X_i, X_j) = 2X_1^2 + 2\mu_2 - X_2^2 - \mu_2$.

Under $H_0$, $\varsigma_0(X_i, X_j | X_i = X_j) = X^2 - 2$.

From the above equation, it is clear that $E[\varsigma_0(X_i, X_j)] = 0$ and $\sigma^2 = E[(\varsigma_0(X_i, X_j))^2] = 20$. \qed

**Corollary 4.1.** Under $H_0$, the limiting distribution of $U_n$ is normal with mean $\hat{\Delta}$.

The variance of $\sqrt{n}(U_n)$ is a function of $\lambda_1, \lambda_2$ and $\lambda_{12}$.

**Proof.** Since the variances of $\sqrt{n}(U_n)$ is very complicated under $H_0$ and since $U_n$ is a function of $U$-statistic, jackknifing would not only reduce the bias, but also enable us to estimate the variance of $V(\sqrt{n}U_n)$.

The estimate of $V(\sqrt{n}U_n)$ is $\hat{V}(\sqrt{n}U_n) = \frac{n}{n-1} \sum_{i=1}^{n} [U_{n,i} - U_n^*]^2$, where,

$U_{n,i} = U_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ and $U_n^* = \frac{1}{n} \sum_{i=1}^{n} U_{n,i}$

Using the results from [6], $\frac{\sqrt{n}(U_n)}{\hat{V}(\sqrt{n}U_n)^{1/2}} \sim N(0,1)$ asymptotically. \qed

5. **Monte Carlo Simulations**

In this section the Monte Carlo null distribution critical points of $\hat{\Delta}$ are simulated based on 10000 generated samples of size $n = 6(1)40$. Table 1 gives the upper percentile points of statistic $\hat{\Delta}$ for different confidence levels 90%, 95%, 98% and 99%. It is clear from Table 1 and Figure 1, that the critical values are increase as the confidence level increases and are almost increases as the sample size increase.
Table 1. Critical Values of the statistic $\hat{\Delta}$

<table>
<thead>
<tr>
<th>n</th>
<th>90%</th>
<th>95%</th>
<th>98%</th>
<th>99%</th>
<th>n</th>
<th>90%</th>
<th>95%</th>
<th>98%</th>
<th>99%</th>
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<td>35.547</td>
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<td>47.839</td>
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<td>62.710</td>
<td>80.864</td>
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<td>63.335</td>
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<td>59.238</td>
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<td>86.869</td>
<td>38</td>
<td>59.836</td>
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<td>58.415</td>
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<td>86.673</td>
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<td>89.678</td>
<td></td>
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</tr>
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</table>

6. Application to Real-Life Data

Here, we present a real life example to illustrate the use of our test statistics $\hat{\Delta}$. We consider the example given by Barlow and Proschan in [1], which is a list of paired first failure time (in hours) of the transmission and the transmission pump on 15 Caterpillar tractors. We use our test to detect whether these failure times follow a bivariate exponential distribution.

Table 2: First failure times of transmission ($X$) and transmission pump ($Y$) on D9G-66A Caterpillar Tractors.

<table>
<thead>
<tr>
<th>Tractor Number</th>
<th>$X$</th>
<th>$Y$</th>
<th>$\min(X,Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1641</td>
<td>850</td>
<td>850</td>
</tr>
</tbody>
</table>
Using the data in Table 2, we obtained, the value of the test statistic \( \hat{\Delta} = 623.4667 \).

Thus \( \frac{\sqrt{n}(U_n)}{[V(\sqrt{n}U_n)]^{\frac{1}{2}}} = 1.219678 \times 10^4 \).

Hence, we reject \( H_0 \) and conclude in favour of BHNBUE. Then we accept \( H_1 \) which shows that the data set has BHNBUE property, but not Bivariate exponential.
7. CONCLUSION

The BHNBUE class of life-time distribution is considered. The moment inequality is derived. A new test statistic for testing BHNBUE class of life-time distributions is proposed based on the moment inequality. Using Monte Carlo Method, critical values of the proposed test is calculated for $n = 6(1)40$ and tabulated. Finally, application to real-life data is carried out.

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