ON FUZZY TOPOLOGICAL BRK-IDEAL

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ABSTRACT. In this article, the notion of fuzzy topological BRK-ideal of a BRK-algebra in a topology is introduced. Some theorems and properties of $f_{\tau}BRKI$ are stated and proved. The epimorphic and into homomorphic inverse images of a $f_{\tau}BRKI$ is also studied well. Also, we introduced a Cartesian product of a $f_{\tau}BRKI$ and studied their properties.

1. INTRODUCTION

Imai and Iseki [3] subjected two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras in the year of 1996. In 1983, the notion of a $BCH$-algebra was introduced by Hu and Li [2], which is a generalization of $BCK$ and $BCI$-algebras. In 2002, a new notion $B$-algebra was introduced by Neggers and Kim [8]. Also a $BF$-algebra and $BG$-algebra was introduced by Walendziak [11] in 2007 and C. B. Kim and H. S. Kim [5], which is a generalization of $B$-algebra. In 2012, R. K. Bandaru [9] introduced BRK-algebra, which is a generalization of $BCK/BCI/BCH/Q/QS/BM$-algebras [4, 6, 7]. In [1], El-Gendy introduced the notion of fuzzy $BRK$-ideal of $BRK$-algebra. S. Sivakumar et al. introduced a topology on $BRK$-algebra [10] and also studied several concepts. In this present paper we introduce a new notion of $f_{\tau}BRKI$ of a $\tau BRK$ Alg. Also study some related properties in a $f_{\tau}BRKI$. At last we introduce the Cartesian product of a $f_{\tau}BRKI$ and their properties.

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2. Preliminaries

Definition 2.1. [9] A BRK-algebra (briefly, BRK Alg) \((I, \ast, 0)\) is a non-empty set \(I\) with a constant \(0\) and a binary operation \(\ast\) satisfying the following axioms:

\[(BRK_1)\ i_1 \ast 0 = i_1,\]
\[(BRK_2)\ (i_1 \ast i_2) \ast i_1 = 0 \ast i_2\]

for any \(i_1, i_2 \in I\). In a BRK Alg I, \(\le\) a partially ordered relation can be defined by \(i_1 \le i_2\) iff \(i_1 \ast i_2 = 0\).

Definition 2.2. [10] Let \((I, \ast, 0)\) be a BRK Alg and \(\tau\) a topology on \(I\). Then \(I = (I, \ast, 0, \tau)\) is called a topological BRK Alg (briefly, \(\tau\)BRK Alg), if “\(\ast\)” is continuous or equivalently, for any \(m, n \in X\) and \(\forall O\) open set of \(m \ast n\), \(\exists\) two open sets \(M\) and \(N\) respectively, such that \(M \ast N\) is a subset of \(O\).

Definition 2.3. [10] Let \(I\) be a \(\tau\)BRK Alg and \(D\) be a subset of \(I\), then \(D\) is called a \(\tau\)BRK-ideal (briefly, \(\tau\)BRKI) of \(I\), if for any \(i_{11}, i_{22} \in I\):

(i) \(0 \in D\),
(ii) \(0 \ast (i_{11} \ast i_{22}) \in D\) and \(0 \ast i_{22} \in D\) imply \(i_{11}, i_{22} \in I\).

Definition 2.4. [1] Let \(I\) be a set. A function \(\mu_I : I \to [0, 1]\) where \(\mu_I\) a fuzzy set in \(I\).

Definition 2.5. [1] Let \((I, \ast, 0)\) be a BRK Alg. A fuzzy set \(\mu_I\) in \(I\) is called a fuzzy BRK-ideal (briefly, \(f\)BRKI) of \(I\) if

\[(BRKFI_1)\ \mu_I(0) \geq \mu_I(i_1),\]
\[(BRKFI_2)\ \mu_I(0 \ast i_1) \geq \min\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}, \text{ for all } i_1, i_2 \in I.\]

3. Fuzzy \(\tau\)BRK-Ideal

Definition 3.1. Let \((I, \ast, 0, \tau)\) be a \(\tau\)BRK Alg. A fuzzy set \(\mu_I\) in \(I\) is called a fuzzy topological BRK-ideal (briefly, \(f\tau\)BRKI) of \(I\) if

\[(3.1)\ \mu_I(0) \geq \mu_I(i_1),\]
\[(3.2)\ \mu_I(0 \ast i_1) \geq \min\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}, \text{ for all } i_1, i_2 \in I.\]
Definition 3.2. Let $(I, *, 0, \tau)$ be a $\tau$BRK Alg. A fuzzy set $\mu_I$ in $I$ is called an Anti fuzzy topological $\tau$BRK-ideal (briefly, $A f\tau$BRK $I$) of $I$ if

\begin{align}
\mu_I(0) &\leq \mu_I(i_1), \\
\mu_I(0 \ast i_1) &\leq \max\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}, \text{ for all } i_1, i_2 \in I.
\end{align}

Example 1. Let $(I = \{0, a_1, b_1, c_1\}, *, 0)$ be a $BRK$ Alg defined by

\[
\begin{array}{cccc}
* & 0 & a_1 & b_1 & c_1 \\
0 & 0 & 0 & b_1 & b_1 \\
a_1 & a_1 & 0 & b_1 & b_1 \\
b_1 & b_1 & b_1 & 0 & 0 \\
c_1 & c_1 & c_1 & a_1 & 0 \\
\end{array}
\]

Define a topology $\tau = \{\phi, I, \{b_1\}, \{b_1, c_1\}, \{0, a_1\}, \{0, a_1, b_1\}, \{0, a_1, c_1\}\}$ is a $\tau$BRK Alg. Now define $\mu_I : I \rightarrow [0, 1]$ by $\mu_I(0) = K_1, \mu_I(a_1) = \mu_I(b_1) = \mu_I(c_1) = K_2$, where $K_1, K_2 \in [0, 1]$ with $K_1 > K_2$ gives that $\mu_I$ is an $f\tau$BRKI.

Proposition 3.1. Let $\mu_I$ be an $f\tau$BRKI of $\tau$BRK Alg $I$ and if $i_1 \geq i_2$, then $\mu_I(0 \ast i_1) \geq \mu_I(0 \ast i_2)$, $\forall i_1, i_2 \in I$.

Proof. Let $\mu_I$ be an $f\tau$BRKI of a $\tau$BRK Alg $I$. For any $i_1, i_2 \in I$ such that $i_1 \geq i_2$. Since $i_1 \geq i_2$, then $i_1 \ast i_2 = 0$.

\[
\mu_I(0 \ast i_1) \geq \min\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}
\]

\[
= \min\{\mu_I(0 \ast 0), \mu_I(0 \ast i_2)\}
\]

\[
= \min\{\mu_I(0), \mu_I(0 \ast i_2)\}
\]

\[
= \mu_I(0 \ast i_2).
\]

Hence $\mu_I(0 \ast i_1) \geq \mu_I(0 \ast i_2)$. $\square$

Theorem 3.1. A fuzzy subset $\mu_I$ of a $\tau$BRK Alg $I$ is a $A f\tau$BRK $I$ of $I$ iff $\mu_I^c$ is an $f\tau$BRKI of $I$.

Proof. Let $\mu_I$ be a $A f\tau$BRK $I$ of a $\tau$BRK Alg $I$, and let $i_1, i_2 \in I$. Then Since $\mu_I(0) \leq \mu_I(i_1)$ then

\[
1 - \mu_I(0) \geq 1 - \mu_I(i_1)
\]

\[
(3.5)
\]

\[
\mu_I^c(0) \geq \mu_I^c(i_1).
\]

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Further,
\[
\mu_I(0 \ast i_1) \leq \max\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}
\]
\[
1 - \mu_I(0 \ast i_1) \geq 1 - \max\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}
\]
\[
\mu_I^c(0 \ast i_1) \geq \min\{1 - \mu_I(0 \ast (i_1 \ast i_2)), 1 - \mu_I(0 \ast i_2)\}
\]
(3.6)
\[
\mu_I^c(0 \ast i_1) \geq \min\{\mu_I^c(0 \ast (i_1 \ast i_2)), \mu_I^c(0 \ast i_2)\}
\]
So, \(\mu_I^c\) is an \(f \tau BRKI\) of \(I\).

Now let \(\mu_I^c\) is an \(f \tau BRKI\) of a \(\tau BRK\) Alg \(I\), and let \(i_3, i_4 \in I\). Then Since \(\mu_I^c(0) \geq \mu_I^c(i_3)\) then
\[
1 - \mu_I^c(0) \leq 1 - \mu_I^c(i_3)
\]
(3.7)
\[
\mu_I(0) \leq \mu_I(i_3).
\]
So,
\[
\mu_I^c(0 \ast i_3) \geq \min\{\mu_I^c(0 \ast (i_3 \ast i_4)), \mu_I^c(0 \ast i_4)\}
\]
\[
1 - \mu_I^c(0 \ast i_3) \leq 1 - \min\{\mu_I^c(0 \ast (i_3 \ast i_4)), \mu_I^c(0 \ast i_4)\}
\]
\[
\mu_I(0 \ast i_3) \leq \max\{1 - \mu_I^c(0 \ast (i_3 \ast i_4)), 1 - \mu_I^c(0 \ast i_4)\}
\]
(3.8)
\[
\mu_I(0 \ast i_3) \leq \max\{\mu_I(0 \ast (i_3 \ast i_4)), \mu_I(0 \ast i_4)\}
\]
Therefore, \(\mu_I\) is a \(Af \tau BRK I\) of a \(\tau BRK\) Alg \(I\). \(\square\)

**Theorem 3.2.** Let \(\mu_I\) be an \(f \tau BRKI\) of \(\tau BRK\) Alg \(I\). Then \(\mu_I = \{i_1 \in I | \mu_I(0 \ast i_1) = \mu_I(0)\}\) is a \(\tau BRK I\).

**Proof.** Clearly \(0 \in I_{\mu_I}\). Let \(i_1, i_2 \in I_{\mu_I}\) be such that \((0 \ast (i_1 \ast i_2)) \in I_{\mu_I}\) and \(0 \ast i_2 \in I_{\mu_I}\). Then \(\mu_I(0 \ast (i_1 \ast i_2)) = \mu_I(0 \ast i_2) = \mu_I(0)\). It follows that
\[
\mu_I(0 \ast i_1) \geq \min\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}
\]
\[
\mu_I(0 \ast i_1) \geq \min\{\mu_I(0), \mu_I(0)\}
\]
\[
\mu_I(0 \ast i_1) \geq \mu_I(0).
\]
So, by combining with Definition 3.1, we get that \(\mu_I(0 \ast i_1) = \mu_I(0)\) and hence \(0 \ast i_1 \in I_{\mu_I}\). \(\square\)
Definition 3.3. Let \((I, \ast, 0, \tau)\) and \((J, \ast', 0', \tau)\) be \(\tau\)BRK Alg's. A mapping \(h : I \rightarrow J\) is said to be a homomorphism of a \(\tau\)BRK Alg if \(h(i_1 \ast i_2) = h(i_1) \ast' h(i_2)\), \(\forall i_1, i_2 \in I\).

Definition 3.4. Let a map \(h : I \rightarrow J\). If \(\mu_I^*\) is a fuzzy subset of \(J\), then the fuzzy subset defined by \(\mu_I^*(h(i_1)) = \mu_I(i_1) \forall i_1 \in I\) is said to be the inverse image of \(\mu_I^*\) under \(h\).

Theorem 3.3. The epimorphic image of an \(f\tau\)BRKI is also an \(f\tau\)BRKI.

Proof. Let \(h : I \rightarrow J\) be an epimorphism of \(\tau\)BRK Alg’s \((I, \ast, 0, \tau)\) and \((J, \ast', 0', \tau)\). Consider that \(\beta\) is an \(f\tau\)BRKI of \(I\) and \(\mu_I\) is the image of \(\beta\) under \(h\). Let \(j_1 \in J\). Then \(\exists i_1 \in I\) such that \(h(i_1) = j_1\). Then

\[\mu_I(j_1) = \mu_I(h(i_1)) = \beta(i_1) \leq \beta(0) = \mu_I(h(0)) = \mu_I(0').\]

Let \(i_1', j_1' \in J\). Then \(\exists i_1, j_1 \in I \ni h(i_1) = i_1' \& h(j_1) = j_1'\). It follows that

\[\mu_I(0' \ast' i_1') = \mu_I(h(0 \ast i_1)) = \beta(0 \ast i_1) \leq \beta(0 \ast (i_1 \ast j_1)) = \min\{\beta(0 \ast (i_1 \ast j_1)), \beta(0 \ast j_1)\} = \min\{\mu_I(h(0 \ast (i_1 \ast j_1))), \mu_I(h(0 \ast j_1))\} = \min\{\mu_I(h(0) \ast' (h(i_1) \ast' h(j_1))), \mu_I(h(0) \ast' h(j_1))\} = \min\{\mu_I(0' \ast' (i_1' \ast' j_1')), \mu_I(0' \ast' j_1')\}.

Hence \(\mu_I\) is an \(f\tau\)BRKI of \(J\). \(\square\)

Theorem 3.4. The into homomorphic inverse image of an \(f\tau\)BRKI is also an \(f\tau\)BRKI.

Proof. Let \(h : I \rightarrow J\) be an into homomorphism of \(\tau\)BRK Alg’s \((I, \ast, 0, \tau)\), \((J, \ast', 0', \tau)\). And \(\mu_I^*\) is an \(f\tau\)BRKI of \(J\) and \(\mu_I\) is the image of \(\mu_I^*\) under \(h\). By definition 3.4 we find that \(\mu_I^*(h(i_1)) = \mu_I(i_1)\), for all \(i_1 \in I\), since \(\mu_I^*\) is an \(f\tau\)BRKI of \(J\), then \(\mu_I^*(0') \geq \mu_I^*(h(i_1)) \forall i_1 \in I\).

So that (3.7) holds, since \(\mu_I(0) = \mu_I^*(h(0)) = \mu_I^*(0') \geq \mu_I^*(h(i_1)) = \mu_I(i_1)\). For all \(i_1, i_2 \in I\), we have

\[\mu_I(0 \ast i_1) = \mu_I^*(h(0 \ast i_1)) = \mu_I^*(h(0) \ast' h(i_1)) \geq \min\{\mu_I^*(h(0) \ast' h(i_1)), \mu_I^*(h(0) \ast' h(i_2))\} = \min\{\mu_I^*(h(0 \ast (i_1 \ast i_2))), \mu_I^*(h(0 \ast i_2))\} = \min\{\mu_I(0 \ast (i_1 \ast i_2)), \mu_I(0 \ast i_2)\}.
\]
Hence \( \mu_I(0 \star i_1) = \mu_I^*(h(0 \star i_1)) = (\mu_I^* \circ h)(0 \star i_1) \) is an \( f\tau BRKI \) of \( I \). The proof is complete. \ \ \ \Box

4. Cartesian Product of \( f\tau BRK \)-Ideal

Definition 4.1. A \( \mu_I \) be fuzzy relation on any set \( I \) is a fuzzy subset \( \mu_I : I \times I \to [0, 1] \).

Definition 4.2. Let \( \mu_I \) and \( \mu_I^* \) be fuzzy subsets of a set \( I \). The Cartesian product of \( \mu_I \) and \( \mu_I^* \) is defined by \( (\mu_I \times \mu_I^*)(i_1, j_1) = \min\{\mu_I(i_1), \mu_I^*(j_1)\} \forall i_1, j_1 \in I \).

Corollary 4.1. Let \((I, *, 0, \tau)\) and \((J, \star, 0', \tau)\) be \( \tau BRK \) Alg's, we define \( \star \) on \( I \times J \) by for every \((i_3, i_4), (j_3, j_4) \in I \times J\), \((i_3, i_4) \star (j_3, j_4) = (i_3 \star j_3, i_4 \star j_4)\) then \((I \times J, \star, (0, 0'), \tau)\) is a \( \tau BRK \) Alg.

Proof. Let \((I, *, 0, \tau)\) and \((J, \star, 0', \tau)\) be \( \tau BRK \) Alg's (see Definition 3.1). For all \((i_3, i_4), (j_3, j_4) \in I \times J\), then

\[
(i) - (i_3, i_4) \star (0, 0') = (i_3 \star 0, i_4 \star 0') = (i_3, i_4)
\]

\[
(ii) - ((i_3, i_4) \star (j_3, j_4)) \star (i_3, i_4) = (i_3 \star j_3, i_4 \star j_4) \star (i_3, i_4)
\]

\[
= ((i_3 \star j_3) \star i_3, (i_4 \star j_4) \star i_4) = (0 \star j_3, 0' \star j_4).
\]

So, \((I \times J, \star, (0, 0'), \tau)\) is a \( \tau BRK \) Alg. \ \ \ \Box

Theorem 4.1. If \( \mu_I \) and \( \mu_I' \) are \( f\tau BRKI \)'s of \( \tau BRK \) Alg's \( I \), then \( \mu_I \times \mu_I' \) is an \( f\tau BRKI \) of \((I \times I, \star, (0, 0'), \tau)\).

Proof. Let \( i_3, i_3' \in I \times I \). Then

\[
(\mu_I \times \mu_I')(0, 0') = \min\{\mu_I(0), \mu_I'(0')\} \geq \min\{\mu_I(i_3), \mu_I'(i_3')\} = (\mu_I \times \mu_I')(i_3, i_3').
\]

For any \((i_3, i_3'), (i_4, i_4') \in I \times I\) we have

\[
(\mu_I \times \mu_I')(0 \star i_3, 0' \star i_3') = \min\{\mu_I(0 \star i_3), \mu_I'(0' \star i_3')\}
\]

\[
= \min\{\min\{\mu_I(0 \star i_3), \mu_I(0 \star i_3')\}, \min\{\mu_I'(0' \star i_3), \mu_I'(0' \star i_3')\}\}
\]

\[
= \min\{\min\{\mu_I(0 \star i_3), \mu_I'(0' \star i_3')\}, \min\{\mu_I(0 \star i_3), \mu_I'(0' \star i_3')\}\}
\]

\[
= \min\{\mu_I(0 \star i_3), \mu_I'(0' \star i_3')\} = (\mu_I \times \beta)((0, 0') \star ((i_3, i_3') \star (i_4, i_4'))), (\mu_I \times \mu_I')((0, 0') \star (i_4, i_4'))\}
\]

Hence \( \mu_I \times \mu_I' \) is a \( f\tau BRKI \) of \((I \times I, \star, (0, 0'), \tau)\). \ \ \ \Box
**Definition 4.3.** If \( \zeta \) is a fuzzy subset of a set \( I \), the strongest fuzzy relation on \( I \) that is a fuzzy relation on \( \zeta \) is \( \mu_\zeta \) given by \( \mu_\zeta(i_1, i_2) = \min\{\zeta(i_1), \zeta(i_2)\} \) \( \forall i_1, i_2 \in I \).

**Proposition 4.1.** For a fuzzy subset \( \zeta \) of a \( \tau\text{BRK Alg} \) \( I \), let \( \mu_\zeta \) be the strongest fuzzy relation on \( I \). If \( \mu_\zeta \) is an \( f\tau\text{BRKI} \) of \( (I \times I; \star, (0, 0)) \), then \( \zeta(0) \geq \zeta(i_1) \) for all \( i_1 \in I \).

**Proof.** Since \( \mu_\zeta \) is a \( f\tau\text{BRKI} \) of \( I \times I \), it follows from (3.5) that \( \mu_\zeta(0, 0) \geq \mu_\zeta(i_1, i_1) \). So that \( \mu_\zeta(0, 0) = \min\{\zeta(0), \zeta(0)\} \geq \max\{\zeta(i_1), \zeta(i_1)\} = \mu_\zeta(i_1, i_1) \). This implies that \( \zeta(0) \geq \zeta(i_1) \). \( \Box \)

**Theorem 4.2.** Let \( \zeta \) be a fuzzy subset of a \( \tau\text{BRK Alg} \) \( I \) and \( \mu_\zeta \) be the strongest fuzzy relation on \( I \). If \( \zeta \) is a \( f\tau\text{BRKI} \) of \( I \) then \( \mu_\zeta \) is a \( f\tau\text{BRKI} \) of \( (I \times I; \star, (0, 0'), \tau) \).

**Proof.** Suppose that, \( \zeta \) is a fuzzy subset of a \( f\tau\text{BRKI} \) \( I \) and \( \mu_\zeta \) is the strongest fuzzy relation on \( I \). Then \( \mu_\zeta(0, 0') = \min\{\zeta(0), \zeta(0')\} \geq \min\{\zeta(i_1), \beta(j_1)\} = \mu_\zeta(i_1, j_1) \forall (i_1, j_1) \in I \times I \).

For all \( (i_1, i_1'), (j_1, j_1') \in I \times I \), we get that
\[
\mu_\zeta((0, 0') \star (i_1, i_1')) = \mu_\zeta(0 \star i_1, 0' \star i_1') = \min\{\beta(0 \star i_1), \beta(0' \star i_1')\} \\
\geq \min\{\min\{\beta(0 \star (i_1 \star j_1)), \beta(0 \star j_1)\}, \min\{\zeta(0' \star (i_1' \star j_1')), \zeta(0' \star j_1')\}\} \\
= \min\{\min\{\zeta(0 \star (i_1 \star j_1)), \beta(0' \star (i_1' \star j_1'))\}, \min\{\beta(0 \star j_1), \beta(0' \star j_1')\}\} \\
= \min\{\mu_\zeta(0 \star (i_1 \star j_1), 0' \star (i_1' \star j_1')), \mu_\zeta(0 \star j_1, 0' \star j_1')\} \\
= \min\{\mu_\zeta((0, 0') \star ((i_1, i_1') \star (j_1, j_1'))), \mu_\zeta((0, 0') \star (j_1, j_1'))\}.
\]
Hence \( \mu_\zeta \) is a \( f\tau\text{BRKI} \) of \( (I \times I; \star, (0, 0'), \tau) \). \( \Box \)

5. Conclusion

In this paper, the \( f\tau\text{BRKI} \) concept of \( \tau\text{BRK Alg} \) was introduced and studied their properties. The epimorphic and into homomorphic inverse images of a \( f\tau\text{BRKI} \) are also discussed and studied well. The \( f\tau\text{BRKI} \) of a cartesian product was also discussed in this work.

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