INTRODUCTION TO OPEN HUB POLYNOMIAL OF GRAPHS

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ABSTRACT. In this paper we introduce the open hub polynomial of a connected graph $G$. The open hub polynomial of a connected graph $G$ of order $n$ is the polynomial $H_O(G)(x) = \sum_{i=0}^{n} h_O(G, i)x^i$ where $h_O(G, i)$ denotes the number of open hub sets of $G$ of cardinality $i$ and $h_O(G)$ is the open hub number of $G$. We obtain the open hub polynomial of some special classes of graphs. Also we obtain open hub polynomial of join of two graphs.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite ordered graph with no loops and no multiple edges. For graph theoretic terminology we refer [1]. All the graphs considered in this paper are connected ,unless otherwise stated.The concept of hub set was Introduced by M. Walsh [2] . A subset $H$ of $V$ is called a hub set of $G$ if for any two distinct vertices $u, v \in V - H$, either $u$ and $v$ are adjacent or there exists a $u-v$ path $P$ in $G$ such that all the internal vertices of $P$ are in $H$. The minimum cardinality of a hub set of $G$ is called the hub number of $G$ and is denoted by $h(G)$. In [3] we have defined the concept of open hub set of a Graph $G$ as follows.

**Definition 1.1.** A hub set $H$ of a graph $G$ is called an open hub set if the induced sub graph , $< H >$ has no isolated vertices. The minimum cardinality of an open hub set of $G$ is called the open hub number of $G$ and is denoted by $h_O(G)$.

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In this paper we introduce the open hub polynomial of a Graph $G$.

**Definition 1.2.** The open hub polynomial of a graph $G$ of order $n$ is the polynomial

$$H_O G(x) = \sum_{i=h_O(G)} H_O(G, i) x^i$$

where $H_O(G, i)$ denotes the number of open hub sets of cardinality $i$.

### 2. Main Results

From the very definition of open hub polynomial we obtain the following result.

**Theorem 2.1.** Let $G$ be a connected graph of order $n$, then

1. $h_O(G, n) = 1$
2. $h_O(G, i) = 0$ if and only if $i \leq h_O(G) - 1$ or $i \geq n + 1$.
3. If $G_1$ is any sub graph of $G$, then $\deg(H_O G(x)) \geq \deg(H_O G_1(x))$.
4. $0$ is a root of $H_O G(x)$ of multiplicity $h_O(G)$ for all graph $G$.

Now we will find the open hub polynomial of some well known graphs. For the complete graph $K_n$, $h(K_n) = 0$. But $h_O(G) \geq 2$ for any graph $G$. As every two element sub sets of vertex set of $K_n$ is an open hub set we have $h_O(K_n) = 2$. Hence $H_O K_n(x) = (1 + x)^n - 1 - nx$.

**Theorem 2.2.** The open hub polynomial of the star graph $K_{1,n}$, $n \geq 3$ is

$$H_O K_{1,n}(x) = x[(1 + x)^n - 1].$$

**Proof.** Let $u$ be the central vertex of $K_{1,n}$ and $u_1, u_2, \ldots, u_n$ are the pendent vertices. Clearly, $h_O(K_{1,n}) = 2$. For $2 \leq i \leq n$, every open hub set of cardinality $i$ must include the central vertex $u$, hence the number of open hub sets of cardinality $i$ is $\binom{n}{i - 1}$, $2 \leq i \leq n$, so that $H_O K_{1,n}(x) = x[(1 + x)^n - 1]$.

**Theorem 2.3.** The open hub polynomial of $K_{2,n}$ is

$$H_O K_{2,n}(x) = x(x + 2)[(1 + x)^n - 1].$$

**Proof.** Let $\{v_1, v_2\}, \{u_1, u_2, \ldots, u_n\}$ are the bipartition of vertex set of $K_{2,n}$. Clearly, $h_O(K_{2,n}) = 2$. For $2 \leq i \leq n$ there are $\binom{n}{i}$ open hub sets of cardinality $i$, which contains $v_1$ but not $v_2$, $\binom{n}{i - 1}$ open hub sets of cardinality $i$, which contains $v_2$ but not $v_1$, $\binom{n}{i - 2}$ open hub sets of cardinality $i$, which contain both $v_1$ and $v_2$. The sets, $\{v_1, u_1, u_2, \ldots, u_n\}$, $\{v_2, u_1, u_2, \ldots, u_n\}$, $\{v_1, v_2, u_1, u_2, \ldots, u_{n-1}\}$, $\{v_1, v_2, u_1, u_2, \ldots, u_{n-2}, u_n\}$, $\{v_1, v_2, u_2, \ldots, u_n\}$ are the open hub set of cardinality $n + 1$. 


Hence \( H_O K_{2,n}(x) = 2\binom{n}{1}x^2 + (2\binom{n}{2})x^3 + (2\binom{n}{3} + \binom{n}{2})x^4 + \cdots + (2\binom{n-1}{n-2} + \binom{n}{2})x^n + ((\binom{n}{3}+2)x^{n+1} + x^{n+2} = x(x+2)((1+x)^n - 1). \quad \square \)

Next we find the open hub polynomial of the double star graph. A double star graph \( S_{m,n} \) is a tree obtained from the graph \( K_2 \) with two vertices \( u \) and \( v \) by attaching \( m \) pendant edges in \( u \) and \( n \) pendant edges in \( v \).

**Theorem 2.4.** The open hub polynomial of the double star \( S_{m,n} \) is \( H_OS_{m,n}(x) = x^2(1+x)^{m+n}. \)

**Proof.** Let \( U = \{u_1, u_2, \ldots, u_m\}, V = \{v_1, v_2, \ldots, v_n\} \) and \( \{u, v\} \) be the vertices of \( S_{m,n} \) such that \( u \) and \( v \) are adjacent, every vertices in \( U \) are adjacent to \( u \) and every vertices in \( V \) are adjacent to \( v \). Clearly the open hub number of \( S_{m,n} \) is two and \( \{u, v\} \) is the only open hub set of cardinality 2. For \( 3 \leq i \leq m+n+2 \), every open hub set must contain both the vertices \( u \) and \( v \). An open hub set of cardinality \( i \) must contain \( i-2 \) vertices from \( \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \). Hence there are \( \binom{m+n}{i-2} \) such open hub sets. Hence \( H_OS_{m,n}(x) = x^2(1+x)^{m+n}. \quad \square \)

Next we find the open hub polynomial of some graph construction. A lollipop graph \( L_{(m,n)} \) is the graph obtained by joining the complete graph \( K_m \) to a path graph \( P_n \) with a bridge.

**Theorem 2.5.** The open hub polynomial of the lollipop graph \( L_{(m,1)} \) is \( H_O L_{(m,1)}(x) = x[(1+x)^m - 1] + x^{m-1}. \)

**Proof.** Let \( v_1, v_2, \ldots, v_m \) be the vertices of the complete graph and let \( v_1 \) is adjacent to \( v \). Clearly, \( h_O(L_{(m,1)}) = 2 \). Then \( \{v_1, v\}, \{v_1, v_2\}, \ldots, \{v_1, v_m\} \) are the open hub sets of cardinality two. For \( 3 \leq r \leq m-2 \), as every open hub set must contain \( v_1 \), number of open hub sets, which contains \( v \), of cardinality \( r \) is equal to number of sub sets of \( \{v_2, v_3, \ldots, v_m\} \) having cardinality \( r-2 \) and the number of open hub sets, which does not contain \( v \), of cardinality \( r \) is equal to number of sub sets of \( \{v_2, v_3, \ldots, v_m\} \) having cardinality \( r-1 \). For \( r = m-1 \), number of open hub sets which contain both \( v \) and \( v_1 \) is equal to number of sub sets of \( \{v_2, v_3, \ldots, v_m\} \) of cardinality \( m-3 \).

Also \( \{v_1, v_2, \ldots, v_{m-1}\}, \{v_1, v_2, \ldots, v_{m-2}, v_m\}, \ldots, \{v_2, v_3, \ldots, v_m\} \) are open hub sets. The sets \( \{v_1, v_2, \ldots, v_m\}, \{v, v_1, v_2, \ldots, v_m\}, \ldots, \{v, v_1, v_2, \ldots, v_{m-1}\} \) are the open hub sets of cardinality \( m \). Hence \( H_O L_{(m,1)}(x) = x[(1+x)^m - 1] + x^{m-1}. \quad \square \)
Theorem 2.6. The open hub polynomial of the lollipop graph $L_{(3,n)}, n \neq 1, 2$ is, $H_O L_{(3,n)}(x) = (n - 1)x^n + \binom{(n^2 + n + 1)}{2}x^{n+1} + (n + 2)x^{n+2} + x^{n+3}$.

Proof. Let $v_1, v_2, v_3$ be the vertices of the complete graph $K_3$ and $u_1, u_2, \ldots, u_n$ be the vertices of the path graph $P_n$. Let $v_1$ be adjacent $u_1$. Then $h_O(L_{(3,n)}) = n$.

Then a hub set of cardinality $n$ is a sub set of $V(P_n) \cup \{v_1\}$ of cardinality $n$. Among these $n + 1$ hub sets only $\{v_1, u_1, u_2, \ldots, u_{n-2}, u_n\}$ and $\{v_1, u_2, u_3, \ldots, u_n\}$ are not open hub sets. All sub sets of $V(L_{(3,n)})$ of cardinality $n + 1$ is a hub set. Let $v$ be a vertex of degree 2 in $L_{(3,n)}$. Let the open neighbourhood of $v$ is, $N(v) = \{v', v''\}$. Then $V(L_{(3,n)}) - \{v', v''\}$ is not an open hub set. Number of such hub sets $=$ number of vertices of degree 2. Again $V(L_{(3,n)}) - \{v, u_{n-1}\}$ where $v \in \{v_1, v_2, v_3, u_1, u_2, \ldots, u_{n-4}, u_{n-2}\}$ is also not an open hub set. Hence $h_{O,L(3,n),n+1} = \binom{n+3}{2} - (n + 1) - n = \frac{n^2 + n + 4}{2}$. Any subset of cardinality $(n+2)$ which contains the vertex $u_n$ is an open hub set. Hence the result follows.

Remark 2.1.

1. The open hub polynomial of $L_{(3,1)}$ is $H_O L_{(3,1)}(x) = 4x^2 + 3x^3 + x^4$.
2. The open hub polynomial of $L_{(3,2)}$ is $H_O L_{(3,2)}(x) = 2x^2 + 4x^3 + 4x^4 + x^5$.

The Dutch windmill graph $D_3^m$ is the graph obtained by selecting one vertex in each of $m$ triangles and identifying them.

Theorem 2.7. The open hub polynomial of the Dutch windmill graph $D_3^m$ is $H_O D_3^m(x) = x[(1 + x)^{2m} - 1] + mx^{2m-2} + x^{2m}, m \geq 2$.

Proof. Let $v$ be the central vertex and $v_1, v_2, \ldots, v_{2m}$ are the other vertices of $D_3^m$ so that $v_{2i-1}v_{2i}$ forms a triangle for $1 \leq i \leq m$. For $2 \leq i \leq 2m - 1, i \neq 2m - 2$, every open hub set must contain the vertex $v$ and every sub sets containing $v$ must be an open hub set. Therefore number of open hub sets of cardinality $i$ is equal to number of sub sets of $\{v_1, v_2, \ldots, v_{2m}\}$ of cardinality $i - 1$. For $i = 2m - 2$ there are $\binom{2m}{2m-3}$ open hub sets which contains $v$ and there are $m$ open hub sets which does not contains $v$, $\{v_1, v_2, \ldots, v_{n-3}, v_{n-2}\}, \{v_1, v_2, \ldots, v_{n-4}, v_{n-1}, v_n\}$, $\ldots, \{v_3, v_4, \ldots, v_{n-1}, v_n\}$. For $i = 2m$ there are $\binom{2m-1}{2m-1}$ open hub sets which contains $v$ and one open hub set $\{v_1, v_2, \ldots, v_{2m}\}$ which does not contains $v$.

Therefore $H_O D_3^m(x) = x[(1 + x)^{2m} - 1] + mx^{2m-2} + x^{2m}, m \geq 2$.

The $(m, n)$- tadpole graph $T_{m,n}$ is the graph obtained by identifying a vertex $v_k$ of the cycle graph $C_m$ with an end vertex of the path graph $P_{n+1}$. Here we find the open hub polynomial of the tadpole graph $T_{4,n-1}$.
Theorem 2.8. The open hub polynomial of the tadpole graph $T_{4,n-1}$, $n \neq 2,3$ is $H_0T_{4,n-1}(x) = (2n - 3)x^n + (n^2 + n + 6)x^{n+1} + (n + 2)x^{n+2} + x^{n+3}$.

Proof. Let $v_1, v_2, v_3, v_4$ are the vertices of the cycle $C_3$ and let $u_1, u_2, \ldots, u_n$ are the vertices of the path graph $P_n$. Let $v_4$ is identified with $u_1$. Clearly $h_0(T_{4,n-1}) = n$. Then the subsets of $V(P_n)$ of cardinality $n - 1$, which contains both $u_1$ and $u_{n-1}$, union with $\{v_i\}, i = 1, 3$ are open hub sets of cardinality $n$. The set $\{u_1, u_2, \ldots, u_n\}$ is also an open hub set. Hence $h_0(T_{4,n-1}, n) = 2n - 3$. Any subset of $V(T_{4,n-1})$ of cardinality $n + 1$ is a hub set. Among these hub sets, a set which contains $u_n$ but not contains $u_{n-1}$ is not an open hub set. Now $\{v_1, v_2, v_3\} \cup V(P_n) - \{u_{k-1}, u_{k+1}\}, k = 2, 3, \ldots, n - 3, n - 1$ are not open hub sets. The sets $\{v_1, v_3, u_2, \ldots, u_n\}$ and $\{v_2, u_1, u_2, \ldots, u_n\}$ are also not open hub sets. Hence $h_0(T_{4,n-1}, n + 1) = \left(\frac{n^2}{3} + \frac{6}{2}\right) - (n + 1) - (n - 1) = \frac{n^2 + n + 6}{2}$. Any subset of $V(T_{4,n-1})$ of cardinality $n + 2$ which contains $u_{n-1}$ is an open hub set. Hence the result. \[\Box\]

Remark 2.2.

1. The open hub polynomial of $T_{4,1}$ is

$$H_0T_{4,1}(x) = 3x^2 + 6x^3 + 4x^4 + x^5.$$ 

2. The open hub polynomial of $T_{4,2}$ is $H_0T_{4,2}(x) = 3x^3 + 8x^4 + 5x^5 + x^6$.

3. Open Hub Polynomial of Join of Two Graphs

Now we find the open hub polynomial of join of two graphs. The join of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv/u \in V(G_1), v \in V(G_2)\}$.

Theorem 3.1. Let $G_1$ and $G_2$ be two connected graphs of order $n_1$ and $n_2$ respectively and let $G = G_1 + G_2$. Then open hub polynomial of $G$ is $H_0G(x) = [(1 + x)^{n_1} - 1][(1 + x)^{n_2} - 1] + H_0G_1(x) + H_0G_2(x)$.

Proof. If $H$ is an open hub set of $G_j$ for $j = 1, 2$ of cardinality $i$ then $H$ is also an open hub set of $G$ of cardinality $i$. Also for every $H_1 \subset V(G_1)$ and $H_2 \subset V(G_2)$, $H_1 \cup H_2$ is an open hub set of $G$ of cardinality $i = i_1 + i_2$, where $i_j$ is the cardinality of $H_j$ for $j = 1, 2$. Thus $H_0G(x) = [(1 + x)^{n_1} - 1][(1 + x)^{n_2} - 1] + H_0G_1(x) + H_0G_2(x)$. \[\Box\]
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