CHARACTERIZATION OF GENERALIZED COMPLEMENTS OF A GRAPH

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ABSTRACT. For a graph $G(V, E)$, let $P = \{V_1, V_2, V_3, \ldots, V_k\}$ be a partition of vertex set $V(G)$ of order $k \geq 2$. For all $V_i$ and $V_j$ in $P$, $i \neq j$, remove the edges between $V_i$ and $V_j$ in graph $G$ and add the edges between $V_i$ and $V_j$ which are not in $G$. The graph $G_{k}^{P}$ thus obtained is called the $k$–complement of graph $G$ with respect to the partition $P$. For each set $V_r$ in $P$, remove the edges of graph $G$ inside $V_r$ and add the edges of $\overline{G}$ (the complement of $G$) joining the vertices of $V_r$. The graph $G_{k(i)}^{P}$ thus obtained is called the $k(i)$–complement of graph $G$ with respect to the partition $P$. In this paper, we characterize few properties of generalized complements of a graph.

1. INTRODUCTION

Let $G$ be a graph on $n$ vertices and $m$ edges. The complement of a graph $G$, denoted as $\overline{G}$ has the same vertex set as that of $G$, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. If $G$ is isomorphic to $\overline{G}$ then $G$ is said to be self-complementary graph. A graph $G$ is $r$ regular if $\delta(G) = \Delta(G) = r$. If $G$ is any $r$-regular graph then $\overline{G}$ is also $(n - r - 1)$ regular. For all notations and terminologies we refer [1]. E. Sampathkumar et al. in [2, 3] introduced two types of generalized complements of a graph. For completeness we produce these here.

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In this paper, conditions for regularity of generalized complements of a regular graph are found. The generalized complements of a graph isomorphic to its line graph are studied.

**Lemma 1.1.** Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of order $k \geq 2$ of a connected graph $G$ on $n$ vertices. If $d$ is degree of a vertex $v$ in $G$ and $d_i$ is degree if $v$ in $<V_i>$, then degree of $v$ in $G^p_k$ is $n - d + 2d_i - n_i$ where $n_i = |V_i|$. 

**Proof.** Lemma follows by definition of $G^p_k$. □

**Proposition 1.1.** For any $r$-regular graph $G(V, E)$ on $n$ vertices with partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V$ of order $k \geq 2$, $G^p_k$ is regular if (i) and either (ii) or (iii) of the following conditions hold:

(i) $k$ divides $n$ and cardinality of $V_i$ is exactly $\frac{n}{k}$ for every $i = 1, 2, 3, \ldots, k$.

(ii) Vertices in each partite are independent.

(iii) Each vertex $v$ in $<V_i>$ has equal degree.

**Proof.** Let $G(V, E)$ be any $r$-regular graph on $n$ vertices. Let $P = \{V_1, V_2, \ldots, V_k\}$ be partition of $V$ of order $k \geq 2$. Suppose degree of a vertex $v$ in $<V_i>$ is $d_i$ for $i = 1, 2, 3, \ldots, k$ then if conditions (i) and (ii) hold, the vertex $v$ will be adjacent to exactly $n - d - (\frac{n}{k})$ vertices in $G^p_k$. Thus $G^p_k$ is $n - d - (\frac{n}{k})$ regular. On the other hand, if conditions (i) and (ii) hold then the vertex $v$ will be adjacent to exactly $n - d + d_i - (\frac{n}{k})$ vertices in $G^p_k$. Hence the proof. □

**Corollary 1.1.** For any $r$-regular graph $G(V, E)$ on $n$ vertices with partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V$ of order $k \geq 2$,

(1) If $G^p_k$ is $n - d - (\frac{n}{k})$ regular then $G^p_k(i)$ is $d + (\frac{n}{k}) - 1$ regular.

(2) If $G^p_k$ is $n - d + 2d_i - (\frac{n}{k})$ regular then $G^p_k(i)$ is $d - 2d_i + (\frac{n}{k}) - 1$ regular.

**Proof.** Follows from noting that $G^p_k$ and $G^p_k(i)$ are complements of each other. □
**Example 1.** For 2-regular graph on 4 vertices and $k = 2$, let us realize Proposition 1.1 and Corollary 1.1

![Figure 1](image-url)

Here $n = 4, d = 2, d_i = 0, \frac{n}{k} = 2, k = 2$

1.1. *k*(i) – Complement of a graph isomorphic to its line graph.

In this section the $k$(i)–complement of a graph $G$ isomorphic to its line graph are studied. Here for any graph $G$, $L(G)$ denotes line graph, $G^p_k(i)$ denotes $k$(i)–complement of $G$, $|V(G)| = n$ the number of vertices in $G$ and $|E(G)| = e$ number of edges in $G$. $C_n$ the cycle of length $n$, $K_{1,n}$ the star graph and $P_n$ path of length $n - 1$.

**Theorem 1.1.** [1] A connected graph $G$ is isomorphic to its line graph $L(G)$, if and only if $G$ is a cycle.

Martin Aigner [4] showed that there are only two graphs namely $C_5$ and $C_3$ with one pendant edge emanating at each of its vertices have the property that their complement is isomorphic to line graph

**Observation.**

1. Let $G$ be any graph, for an automorphism between $V(G)$ and $V(G^p_k(i))$ one must have $|E(G)| = |V(G)| = |V(G^p_k(i))|$. This implies that $G$ is connected unicyclic graph or $G$ consists of $c$ components each of which are unicyclic.

2. Let $P = \{V_1, V_2, \ldots, V_k\}, k \geq 2$, be a partition of $V$. Then none of $\langle V_i \rangle$ for $i = 1, 2, \ldots, k$ must be complement of any of nine forbidden graphs [1] for line graphs.

**Theorem 1.2.** For any graph $G$ of order $n$ and size $q$, $G^p_k(i)$ is isomorphic to $L(G)$ if any one of the following conditions hold.
(1) $G$ is any cycle $C_n$ and $P = \{V_1, V_2, \ldots, V_n\}$ partition of $V(G)$, with $V_i = 1$ for all $i$.

(2) $G$ is unicyclic with at least one pendant edge attached to a cycle $C_n$ and $P = \{V_1, V_2, \ldots, V_n\}$ partition of $V$ such that each $V_i$ for $i = 1, 2, \ldots, n$, has exactly one vertex $v_i$ of $C_n$ and all the pendant vertices at $v_{i+1}$, all the pendant vertices at $v_q$ are in the partite that contains $v_1$.

(3) $G$ is unicyclic with at least one path attached to a cycle $C_n$ and $P = \{V_1, V_2, \ldots, V_{n-x}\}$ partition of $V$ where $x$ is number of vertices of the path at distance one from vertices of $C_n$ such that each $V_i$ for $i = 1, 2, \ldots, n$, has exactly one vertex $v_i$ of $C_n$ and all the vertices at distance one from $v_{i+1}$ of $C_n$, all the vertices at distance one from $v_q$ belong to the partite that contains $v_1$.

**Proof.** Let $G$ be any graph satisfying condition 1, then by definition of $G^p_i$ isomorphic to $L(G)$. Let $G$ be any graph satisfying condition 2. Let $e_1, e_2, \ldots, e_j$, be the pendant edges at a vertex of $C_n$. Then these edges form a complete subgraph in $L(G)$. This is true at every vertex of $C_n$. Thus $L(G)$ is $C_n$ with complete subgraphs attached at the vertices of $C_n$ for the partition $P = \{V_1, V_2, \ldots, V_n\}$ of $V$ such that each $V_i$ for $i = 1, 2, \ldots, n$ has exactly one vertex $v_i$ of $C_n$ and all the pendant vertices at $v_{i+1}$, all the pendant vertices at $v_q$ are in the partite that contains $v_1$. Then $G^p_i$ is $C_n$ with complete subgraphs attached at the vertices of $C_n$ and isomorphic to $L(G)$.

If $G$ is any graph satisfying condition 3 the result follows similar way. □

**Example 2.** Condition 1.

![Diagram](image)

**Figure 2**

Condition 2.
2. Characterization of Generalized Complements of Euler Graphs

Definition 2.1. A connected graph $G$ is called Eulerian graph if it contains a closed trail containing all the edges of $G$.

Theorem 2.1. If $G(V, E)$ is Eulerian and $P = \{V_1, V_2, \ldots, V_k\}$ partition for $V$ of order $k \geq 2$, such that for at least one $i$, $i = 1, 2, \ldots, k$, $|V_i| = 2$ then $G^p_{k(i)}$ is non Eulerian.

Proof. Let $G(V, E)$ be any Eulerian graph. Then every vertex of $G$ is of even degree. Suppose $P = \{V_1, V_2, \ldots, V_k\}$ partition for $V$ of order $k \geq 2$. Let $V_i$ be a partition having only two vertices say $u$ and $v$. Then in $G^p_{k(i)}$, degree of $u$ and $v$ will be increased by 1 if $u$ and $v$ are not adjacent in $<V_i>$ and degree of $u$ and $v$ will be decreased by 1 if $u$ and $v$ are adjacent in $<V_i>$. Thus in either of the cases $G^p_{k(i)}$ has a vertex of odd degree. Hence $G^p_{k(i)}$ is not Eulerian. □

Theorem 2.2. If $G$ is any Eulerian graph and $P = \{V_1, V_2, \ldots, V_k\}$ partition for $V$ of order $k \geq 2$, then $G^p_{k(i)}$ is Eulerian if all the following conditions hold good.

1. $|V_i|$ is odd for each $i = 1, 2, \ldots, k$.
2. For every vertex $v$ of odd degree in $<V_i>$ there must be odd number of vertices in $V_i$ not adjacent to $v$ for $i = 1, 2, \ldots, k$.
For every vertex \( v \) of even degree in \( < V_i > \) there must be even number of vertices in \( V_i \) not adjacent to \( v \) for \( i = 1, 2, \ldots, k \).

**Proof.** Let \( G \) be any Eulerian graph. Then degree of every vertex of \( G \) is even. Let \( P = \{V_1, V_2, \ldots, V_k\} \) be a partition for \( V \) of order \( k \geq 2 \) for which above three conditions hold. Let vertex \( v \in V_i \) for some \( i = 1, 2, \ldots, k \). Suppose degree of \( v \) in \( G \) is \( d \) and degree of \( v \) in \( < V_i > \) is \( d_i \). If \( d_i \) is odd then there are odd number of vertices say \( n_i \) in \( < V_i > \) which are not adjacent to \( v \). Then degree of \( v \) in \( G^p_k \) is \( d - d_i + n_i \) which is even. On the other hand, if \( d_i \) is even then there are even number of vertices say \( l_i \) in \( < V_i > \) which are not adjacent to \( v \). Then degree of \( v \) in \( G^p_{k(i)} \) is \( d - d_i + l_i \) which is even. This is true for every \( v \) and each \( i = 1, 2, \ldots, k \). Thus every vertex of \( G^p_{k(i)} \) is of even degree. Hence \( G^p_{k(i)} \) is Eulerian. \( \square \)

**References**


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