ACCURATE NEIGHBORHOOD RESOLVING NUMBER OF A GRAPH

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**Abstract.** A neighborhood set of a graph \( G(V, E) \) is a subset \( S \subseteq V \) such that \( G = \bigcup_{v \in S} \langle N[v] \rangle \), where \( N[v] \) is the closed neighborhood of the vertex \( v \). A resolving set of a graph \( G(V, E) \) is a subset \( S \subseteq V \) such that every pair of distinct vertices of \( G \) is resolved by some vertex in \( S \). A neighborhood set of \( G \), which is also a resolving set is called as neighborhood resolving set (\( nr \)-set) of \( G \). An \( nr \)-set \( S \) of \( G \) is called an accurate neighborhood resolving set (\( anr \)-set) of \( G \) if \( S \) has no \( nr \)-set of \( G \) with cardinality of \( S \). In this paper, we determine the minimum cardinality of \( nr \)-sets and \( anr \)-sets of total graph of a cycle and a prism graph.

1. Introduction

The graphs that are considered throughout this paper are finite, simple, connected, nontrivial and undirected. The terms not defined here may be found in [1, 3]. For a graph \( G(V, E) \) and a vertex \( v \in V \), \( N(v) \) denotes the set of all vertices which are adjacent to \( v \) and \( N[v] = N(v) \cup \{v\} \). A subset \( S \) of \( V \) is a neighborhood set (\( n \)-set) of \( G \) if \( \bigcup_{v \in S} \langle N[v] \rangle = G \), where \( \langle N[v] \rangle \) is the sub graph of \( G \) induced by \( N[v] \). The minimum cardinality of an \( n \)-set of \( G \) is called the neighborhood number of \( G \) and is denoted by \( \ln(G) \). Neighborhood number of a graph was first introduced by E. Sampathkumar and Prabha S. Neeralagi [7].

Given a graph \( G \) and a subset \( S \) of the vertex set of \( G \), a vertex \( s \in S \) resolves a pair of vertices \( u, v \in V \), if \( d(u, s) \neq d(v, s) \). A resolving set (\( r \)-set) \( S \) is a subset

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of vertex set of $G$ such that each pair of vertices $u, v \in V(G)$ is resolved by at least one vertex in $S$. If $S = \{s_1, s_2, \ldots, s_k\}$ is a resolving set of $G$, then we can associate a unique vector for each $v \in V(G)$ with respect to $S$ as $\Gamma(v/S) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$, where $d(u, v)$ is the distance between the vertices $u$ and $v$ in $G$. The minimum cardinality of an $r$-set of $G$ is called the resolving number of $G$ and is denoted by $lr(G)$. The concept of resolving number of a graph was first introduced by P. J. Slater [8] and independently by F. Harary and R. A Melter [2].

A subset $S$ of $V$ is called a neighborhood resolving set ($nr$-set) of $G$, if $S$ is both neighborhood set and resolving set of $G$. The minimum cardinality of an $nr$-set is called the neighborhood resolving number of $G$ and is denoted $lnr(G)$. An $nr$-set $S$ of $G$ is called an accurate neighborhood resolving set ($anr$-set) of $G$ if $\overline{S}$ has no $nr$-set of $G$ with cardinality of $S$. The minimum cardinality of an $anr$-set is called the accurate neighborhood resolving number of $G$ and is denoted by $lnr_a(G)$. The concept of $anr$-set was first introduced and studied by Reshma et al. in [6]. For similar works we refer [4, 5, 10, 11].

The total graph $T(G)$ of a graph $G$ is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of $G$ and two vertices are adjacent in $T(G)$, if their corresponding elements are either adjacent or incident in $G$.

We now recall the following results for immediate reference.

**Theorem 1.1** (B. Sooryanarayana, Shreedhar K. and Narahari N. [9]). For a graph $G$, $lr(T(G)) = 2$ if and only if $G$ is a path $P_n$, $n \geq 2$.

**Theorem 1.2** (E. Sampathkumar and P. S. Neeralagi [7]). A set $S$ of vertices of a graph $G$ is an $n$-set if and only if every edge of $\langle V(G) - S \rangle$ belongs to a triangle one of whose vertices belongs to $S$.

If $S$ is an $n$-set of $G$, then we say that an edge $e$ is covered by $S$, if $S$ contains a vertex $s$ such that $s$ is incident with $e$, or $s$ is adjacent to both the end vertices of $e$ in $G$. Also, we note that neighborhood property, resolving property, and neighborhood resolving property are all super hereditary.

**Corollary 1.1** (E. Sampathkumar and P. S. Neeralagi [7]). A set $S$ is an $n$-set of a triangular free graph $G$ if and only if $\overline{S}$ is totally disconnected.

**Observation 1.1.** For any graph $G$, as every $nr$-set is also an $n$-set and an $r$-set of $G$, it follows that $lnr(G) \geq ln(G)$ and $lnr(G) \geq lr(G)$. 
Observation 1.2. For any graph $G$, as every anr-set is also an nr-set, an r-set and an n-set of $G$, it follows that $\lnr_a(G) \geq \lnr(G)$, $\lnr_a(G) \geq \lr(G)$ and $\lnr_a \geq \ln(G)$.

2. Total graph of a cycle

Throughout this section, the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ of the total graph $T(C_n)$ corresponds to the vertices of the cycle $C_n$, and the vertices $e_0, e_1, \ldots, e_{n-1}$ of $T(C_n)$ corresponds to the edge of $C_n$ with $e_i = v_iv_{i+1(\mod n)}$ for each $i$, $0 \leq i \leq n-1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The total graph of the cycle $C_6$.}
\end{figure}

Theorem 2.1. For any integer $n \geq 3$, $\lnr(T(C_n)) = \begin{cases} 3, & \text{if } n = 3. \\ \lceil \frac{2n}{3} \rceil, & \text{if } n \geq 4. \end{cases}$

Proof. Consider the graph $G = T(C_n)$ on $2n$ vertices.

**Lower bound:** Let $S$ be any nr-set of $G$ and $|S| = k$. Without loss of generality, we take $v_0 \in S$ (due to symmetry). The vertex $v_0$ covers exactly 7 edges namely, $v_0v_1$, $v_0e_0$, $v_0v_{n-1}$, $v_0e_{n-1}$, $v_1e_0$, $v_{n-1}e_{n-1}$ and $e_0e_{n-1}$ as per the criteria of the n-set. While covering these seven edges, to cover the edge $e_0e_1$, the set $S$ should include at least one of the elements in the set $T = \{e_0, e_1, v_1\}$. However, each single element in $T \cap S$ will cover at most 6 new edges of $G$ (since one edge is already covered by $v_0$) and $e_1$ is the only vertex in $S \cap T$ which covers the maximum of six edges. Further, $v_3$ is the vertex which covers maximum of 6 edges while covering the next edge $v_2v_3$. Continuing this way, every vertex in $S - \{v_0\}$ will cover at most 6 edges of $G$. Hence $S$ will cover at most $7 + 6(k-1) = 6k + 1$ edges of $G$. Thus, as the graph $G$ is a 4 regular graph $2n$ vertices, (the number of edges in $G$) $4n \leq 6k + 1$. That is $k \geq \lceil \frac{4n-1}{6} \rceil$. Therefore,
\[ \lnr(G) = \min \{|S| : S \text{ is an } nr\text{-set of } G \} \geq \lceil \frac{4n-1}{6} \rceil = \lceil \frac{2n}{3} \rceil. \] But when \( n = 3 \), by Theorem 1.1, \( \lnr(G) \geq 3 \) and hence by Observation 1.1, \( \lnr(G) \geq 3 \).

**Upper bound:** We show the lower bound obtained above is tight by executing an \( nr\)-set \( S \) of \( G \).

**Case 1:** \( 3 \leq n \leq 6 \).

Consider the sets; \( S_3 = \{v_0, v_1, v_2\}, S_4 = \{v_0, e_1, v_2\}, S_5 = \{v_0, e_1, v_3, e_4\} \) and \( S_6 = \{v_0, e_1, v_3, e_4\}. \) It can be easily verified that \( S_3, S_4, S_5 \) and \( S_6 \) are \( nr\)-sets of \( G \) for \( n = 3, 4, 5, 6 \), respectively.

**Case 2:** \( n \geq 7 \).

Consider the set \( S = \begin{cases} \{v_0, e_1, v_3, e_4, \ldots, v_{n-3}, e_{n-2}\} & \text{if } n \equiv 0 \pmod{3}. \\ \{v_0, e_1, v_3, e_4, \ldots, e_{n-3}, v_{n-1}\} & \text{if } n \equiv 1 \pmod{3}. \\ \{v_0, e_1, v_3, e_4, \ldots, v_{n-2}, e_{n-1}\} & \text{if } n \equiv 2 \pmod{3}. \end{cases} \)

The set \( S \) defined above is an \( n\)-set of \( G \). In fact,

**Subcase 1:** \( n \equiv 0 \pmod{3} \).

In this case, \( v_i, e_{j+1} \in S \), for every \( i \equiv 0 \pmod{3} \) and \( 0 \leq i \leq n - 2 \). The edges: \( v_{3i}v_{3i+1} \pmod{n} \) are covered by \( v_{3i} \in S \); \( v_{3i+1} \pmod{n}v_{3i+2} \pmod{n} \) are covered by \( e_{3i+1} \in S \); \( e_{3i+1}e_{3i+2} \pmod{n} \) and \( e_{3i+1}e_{3i-1} \pmod{n} \) are covered by \( e_{3i+1} \in S \); \( e_{3i-1} \pmod{n}e_{3i} \) are covered by \( v_{3i} \in S \); \( e_{3i}v_{3i+1} \pmod{n} \) and \( e_{3i-1} \pmod{n}v_{3i} \) are covered by \( v_{3i} \in S \); \( e_{3i+1}e_{3i+1} \) are covered by \( e_{3i+1} \in S \); \( e_{3i-1} \pmod{n}v_{3i-1} \pmod{n} \) and \( e_{3i}v_{3i} \) are covered by \( v_{3i} \in S \); and \( e_{3i+1}e_{3i+1} \) are covered by \( e_{3i+1} \in S \). Hence, \( G = \bigcup_{v \in S} \langle \{v\} \rangle \) and \( |S| = |S \cap V(C_n)| + |S \cap E(C_n)| = 2|S \cap V(C_n)| = 2 \left[ 1 + \frac{n-3}{2} \right] = \frac{2n}{3} \).

**Subcase 2:** \( n \equiv 1 \pmod{3} \).

In this case, all the edges of \( G \) between two vertices are covered by the set \( S' = \{v_0, e_1, v_3, e_4, \ldots, v_{n-4}, e_{n-3}\} \) as in the above Subcase 1 except the 3 edges, namely \( v_{n-2}v_{n-1}, e_{n-2}v_{n-1} \) and \( e_{n-2}, e_{n-1} \). These 3 edges are now covered by \( v_{n-1} \in S \). Hence \( S \) is an \( n\)-set with \( |S| = |S'| + 1 = 2 \left[ 1 + \frac{n-4}{2} \right] + 1 = \frac{2n+1}{3} = \lceil \frac{2n}{3} \rceil \).

**Subcase 3:** \( n \equiv 2 \pmod{3} \).

In this case, all the edges of \( G \) between two vertices are covered by the set \( S' = \{v_0, e_1, v_3, e_4, \ldots, e_{n-4}, v_{n-2}\} \) as in Subcase 2 except one edge, namely \( e_{n-2}e_{n-1} \).
This extra edge is covered by $e_{n-1} \in S$. Hence $S$ is an $n$-set and $|S| = 2[1 + \frac{n-2}{3}] = \frac{2n+2}{3} = \lceil \frac{2n}{3} \rceil$.

Now to prove $lnr(G) \leq \lceil \frac{2n}{3} \rceil$, it remains to show that the set $S$ defined above is also an $r$-set. For this, let $S_1 = \{v_0, e_1\}$. Then the vector associated for each vertex of $G$ with respect to $S_1$ is given by

\[
\begin{align*}
\Gamma(u/S_1) &= (i, 2-i) \quad \text{and} \quad \Gamma(e_i/S_1) = (i+1, 1-i) \quad \text{if} \quad i = 0,1 \\
\Gamma(v_i/S_1) &= (i, i-1) \quad \text{and} \quad \Gamma(e_i/S_1) = (i+1, i-1) \quad \text{if} \quad 2 \leq i \leq [n/2] - 1 \\
\Gamma(v_i/S_1) &= (n-i, i-[\frac{n-1}{2}]) \quad \text{and} \quad \Gamma(e_i/S_1) = (n-i, i-[\frac{n-1}{2}]) \quad \text{if} \quad [\frac{n+1}{2}] \leq i \leq [\frac{n+1}{2}] \\
\Gamma(v_i/S_1) &= (n-i, n-i+2) \quad \text{and} \quad \Gamma(e_i/S_1) = (n-i, n-i+1) \quad \text{if} \quad [\frac{n+1}{2}] < i \leq n-1.
\end{align*}
\]

From the above vector, it is easy to see that $\Gamma(u/S_1) = \Gamma(v/S_1)$ if and only if $(u, v) \in T_1 = \{(v_1, e_0), (v_i, \frac{n+1}{2}), (v, \frac{n+1}{2})\}$. But, for each pair $(u, v) \in T$, $|d(u, v_3) - d(v, v_3)| \neq 0$ and hence $v_3$ will resolve $u$ and $v$. Thus, $S_2 = S_1 \cup \{v_3\}$ is an $r$-set of $G$. Further, as $S_2 \subseteq S$ and super hereditary property of resolving sets, the set $S$ is an $r$-set of $G$. Hence the proof.

**Theorem 2.2.** For any integer $n \geq 3$,

\[
lnr_a(T(C_n)) = \begin{cases} 4, & \text{if } n = 3, \\ \floor{\frac{2n}{3}} + 1, & \text{if } n \equiv 1,2 \pmod{3}, \\ \floor{\frac{2n}{3}} + 2, & \text{if } n \equiv 0 \pmod{3}. \end{cases}
\]

**Proof.** Let $G = T(C_n)$, $n \geq 3$ be the graph with $2n$ vertices.

**Lower bound:** Let $S$ be an $anr$-set. Then $S$ is an $nr$-set and $\bar{S}$ has no $nr$-set of cardinality $|S|$. Without loss of generality, we assume $v_0 \in S$. We first see that if $S$ contains all the three vertices of a triangle in $G$, then $\bar{S}$ is not an $n$-set of $G$ for all $n \geq 4$ and hence $\bar{S}$ has no $nr$-set of any cardinality. Therefore, if $n = 3, 4$ and $|S| = 3$, then $\langle S \rangle = C_3$ and $n \geq 4$ (else by symmetry $G - S$ has a subgraph $H$ isomorphic to $\langle S \rangle$ and hence $V(H)$ will be an $nr$-set of $G$). Further, if $n = 4$, without loss of generality, let $S = \{v_0, e_0, v_1\}$. Then the edge $v_2v_3$ is not in any triangle of $G$ with one vertex in $S$, contradiction to Theorem 1.2. So, $lnr_a(G) \geq 4$, for $n = 3, 4$. Let $n \geq 5$ and $S'$ be any minimum $nr$-set of $G$. Then, by Theorem 2.1, $|S'| = \floor{\frac{2n}{3}}$. Further, each element of $S'$ is a corner vertex of the shaded triangle which is in maximum number of unshaded triangles behind it (starting with $v_0 \in S'$) as in Figure 2. Thus, in each minimum $nr$-set $S'$ if $v_i \in S'$ then $e_i \notin S'$. This shows that the set $S''$ obtained by just interchanging $e_i$ and
Figure 2. Optimal choice of an nr-set.

$v_i$ in $S'$ is also an nr-set of $G$. Since $S'' \subseteq \bar{S}$, it follows that $lnr_a(G) > lnr(G)$. Hence $lnr_a(G) \geq \lceil \frac{2n}{3} \rceil + 1$. Further, if any nr-set $S'$ contains a pair $v_i, e_i$ for at most one $i$ and no two adjacent pairs of $C_n$, then it is easy to verify that the set $S''$ containing $x_{i+1}$ for each $x_i$ in $S'$ is an nr-set of $G$. Therefore, $C_3$ should be an induced subgraph of $\langle S \rangle$ for every minimum anr-set $S$. But then, as these three vertices of a triangle in $S$ will be covering exactly 11 edges, to cover the remaining $4n - 11$ edges we need at least $\frac{4n-11}{6}$ vertices in $S$ other than those three which are in a triangle. That is, $4n-11 \leq 6(|S| - 3)$ implies that $|S| \geq \frac{4n+7}{6}$. So, $|S| \geq \lceil \frac{2n}{3} \rceil + 1$ if $n \equiv 0 \pmod{3}$, and $|S| \geq \lceil \frac{2n}{3} \rceil + 2$ if $n \equiv 0 \pmod{3}$.

Upper bound: Here we show that the above lower bound is tight by executing an anr-set of $G$.

Case 1: $n \equiv 0 \pmod{3}$.

When $n = 3$, it is easy to see that the set $S = \{v_0, v_1, v_2, e_1\}$ is an anr-set for $G$. For $n > 3$, let $S_1$ be a minimum nr-set of $G$. Without loss of generality, we assume $v_0 \in S$. Since $S_1$ is a minimum nr-set, as per the above discussion, $e_0, v_1$ are not in $S$. Taking $S = S_1 \cup \{e_0, v_1\}$ and by the super hereditary of nr property, we see that $S$ is an nr-set of $G$. Further, $\bar{S}$ contains none of the vertices of a triangle of $G$. Hence $\bar{S}$ is not an nr-set implies that $S$ is an anr-set with $|S| = |S_1| + 2 = \lceil \frac{2n}{3} \rceil + 2$.

Case 2: $n \equiv 1 \pmod{3}$.

In this case, consider the nr-set $S_1 = \{v_0, e_1, v_3, e_4, \ldots, e_{n-3}, v_{n-1}\}$ (as in the proof of Theorem 2.1). The set $S = S_1 \cup \{e_{n-1}\}$ is then an nr-set and $\langle \{v_0, e_{n-1}, v_{n-1}\} \rangle$ is an induced cycle $C_3$ of $\langle S \rangle$. Hence, $\bar{S}$ is not an nr-set implies that $S$ is an anr-set with $|S| = |S_1| + 1 = \lceil \frac{2n}{3} \rceil + 1$.

Case 3: $n \equiv 2 \pmod{3}$.
In this case, consider the nr-set \( S_1 = \{ v_0, e_1, v_3, e_4, \ldots, v_{n-2}, e_{n-1} \} \) (as in the proof of Theorem 2.1). The set \( S = S_1 \cup \{ v_{n-1} \} \) is then an nr-set and \( \langle \{ v_0, e_{n-1}, v_{n-1} \} \rangle \) is an induced cycle \( C_3 \) of \( \langle S \rangle \). Hence, \( \bar{S} \) is not an nr-set implies that \( S \) is an anr-set with \( |S| = |S_1| + 1 = \lceil \frac{2n}{3} \rceil + 1 \).

3. Prism graph

The Cartesian product of two graphs \( G \) and \( H \), denoted by \( G \square H \), is a graph whose vertex set is \( V(G) \times V(H) \) and two vertices \( (g, h) \) and \( (g', h') \) are adjacent in \( G \square H \) if either \( g = g' \) and \( hh' \in E(H) \), or \( h = h' \) and \( gg' \in E(G) \). Prism graph is the Cartesian product of \( C_n \) and \( P_2 \), denoted by \( C_n \square P_2 \). Throughout this section, we label the vertices of the prism graph, \( C_n \square P_2 \) as \( v_0, v_1, v_2, \ldots, v_{n-1}, u_0, u_1, u_2, \ldots, u_{n-1} \) such that \( v_i \) is adjacent to \( v_{i+1(\text{mod} n)} \) for all \( i, 0 \leq i \leq n-1 \), \( u_i \) is adjacent to \( u_{i+1(\text{mod} n)} \) for all \( i, 0 \leq i \leq n-1 \) and \( v_i \) is adjacent to \( u_i \) for all \( i, 0 \leq i \leq n-1 \).

![Figure 3. Prism graph, \( C_6 \square P_2 \).](image)

**Theorem 3.1.** For any integer \( n \geq 3 \) and a prism graph \( C_n \square P_2 \),

\[
\lnr(C_n \square P_2) = 2 \left\lceil \frac{n}{2} \right\rceil.
\]

**Proof.** Let \( G = C_n \square P_2 \) be a prism graph on \( 2n \) vertices.

**Lower bound:** The graph \( G \) is triangle free and hence for every \( n \)-set \( S \) of \( G \), by Corollary 1.1, its complement \( \bar{S} \) is totally disconnected and vice versa. Thus, the \( nr \)-number of the graph \( G \) is equal to \( |V(G)| - \text{id}(G) \), where \( \text{id}(G) \) is the independent number of \( G \). Let \( S_u = \{ u_i : 0 \leq i \leq n-1 \} \) and \( S_v = \{ v_i : 0 \leq i \leq n-1 \} \). Then \( S_u \) and \( S_v \) are the partition of \( V(G) \). Since \( \langle S_u \rangle \) and
\( \langle S_v \rangle \) are isomorphic to the cycle \( C_n \), each independent set of \( G \) contains at most \( \lfloor \frac{n}{2} \rfloor \) vertices from each of these sets. Therefore, \( \lnr(G) \geq \lnr(G) \geq 2n - 2\lfloor \frac{n}{2} \rfloor = 2 \left( n - \lfloor \frac{n}{2} \rfloor \right) = 2\lceil \frac{n}{2} \rceil \).

**Upper bound:** We show that lower bound obtained above is tight by exacting an \( nr \)-set \( S \). For this, we first consider the set \( T = \{ v_0, v_3 \} \).

**Case 1:** \( 3 \leq n \leq 6 \).

Consider the sets; \( S_3 = \{ v_0, u_0, v_1, u_3 \} \), \( S_4 = \{ v_0, u_1, v_2, u_3 \} \), \( S_5 = \{ v_0, u_0, u_1, v_2, u_3, v_4 \} \) and \( S_6 = \{ v_0, u_1, v_2, u_3, v_4, u_5 \} \). It can be easily verified that \( S_3, S_4, S_5 \) and \( S_6 \) are \( r \)-sets of \( G \) for \( n = 3, 4, 5, 6 \), respectively. Also, for each of the sets \( \tilde{S} \) is independent. Hence they are the desired \( nr \)-sets.

**Case 2:** \( n \geq 7 \).

The vectors associated to each vertex of \( G \) with respect to \( T \) is as below.

\[
\begin{align*}
\Gamma(v_i/T) &= (i, 2 - i) \quad \text{and} \quad \Gamma(u_i/T) = (i + 1, 3 - i) \quad \text{for} \quad 1 \leq i \leq 2 \\
\Gamma(v_i/T) &= (i, i - 2) \quad \text{and} \quad \Gamma(u_i/T) = (i + 1, i - 1) \quad \text{for} \quad 3 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
\Gamma(v_i/T) &= (n - i, i - 2) \quad \text{and} \quad \Gamma(u_i/T) = (n + 1 - i, i - 1) \quad \text{for} \quad \lceil \frac{n}{2} \rceil \leq i \leq \lceil \frac{n}{2} \rceil + 1 \\
\Gamma(v_i/T) &= (n - i, n + 2 - i) \quad \text{and} \quad \Gamma(u_i/T) = (n + 1 - i, n + 3 - i) \quad \text{for} \quad \lceil \frac{n}{2} \rceil + 2 \leq i \leq n - 1
\end{align*}
\]

This shows that, \( T \) will resolve all the pairs of vertices of \( G \) except those in \( H = \{(v_i, u_{i-1}) : 3 \leq i \leq \lfloor \frac{n}{2} \rfloor \} \cup \{(v_i, u_{i+1} \mod n) : \lceil \frac{n}{2} \rceil + 2 \leq i \leq n - 1 \} \). Now for each pair \( (v_i, u_{i+1}) \in H \), \( d(v_i, u_1) = d(u_{i+1}, u_1) + 2 \) and hence \( u_1 \), will resolve all the pairs in \( H \). Thus, the set \( S_1 = T \cup \{ u_1 \} \) is an \( r \)-set for \( G \).

Now consider the set \( S = \left\{ \begin{array}{ll} \{ v_0, u_1, v_2, u_3, \ldots, v_{n-2}, u_{n-1} \} & \text{if } n \text{ is even} \\
\{ v_0, u_1, v_2, u_3, \ldots, v_{n-1}, u_0 \} & \text{if } n \text{ is odd} \end{array} \right. \)

The set \( S_1 \) is a subset of \( S \) and hence by super hereditary of the resolving property, \( S \) is an \( r \)-set. Also, \( \tilde{S} \) is an independent set of \( G \). Hence, by Corollary 1.1, \( S \) is an \( nr \)-set and it contains \( 2 \lceil \frac{n}{2} \rceil \) vertices of \( G \). Hence \( \lnr(G) \leq 2 \lceil \frac{n}{2} \rceil \). Hence the proof.

We now state the following generalized theorem whose proof follows similar to the above theorem.

**Theorem 3.2.** For the integers \( m \geq 1 \) and \( n \geq 3 \),
\[
\lnr(C_n \square P_m) = m \lceil \frac{n}{2} \rceil.
\]
For each odd $n$, every minimum $nr$-set $S$ of the prism $C_n \square P_2$ contains $n + 1$ elements (by Theorem 3.1). Therefore, $|\bar{S}| = 2n - (n + 1) = n - 1 < |S|$, implies that $\bar{S}$ can not have any $nr$-set with cardinality $|S|$. Hence every minimum $nr$-set of $C_n \square P_2$ is also an $anr$-set of $G$ whenever $n$ is odd. But this is not the case when $n$ is even. If $n$ is even, then both $S$ and $\bar{S}$ are independent with $|S| = |\bar{S}| = n$, for every $nr$-set $S$ of $C_n \square P_2$. Therefore, every minimum $anr$-set should contain one more element than in a minimum $nr$-set. We record these in the form of following theorem.

**Corollary 3.1.** For any integer $n \geq 3$ and a prism graph $C_n \square P_2$,

$$lnr_a(C_n \square P_2) = n + 1.$$ 

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