UNITARY DIVISOR ADDITION CAYLEY GRAPHS

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ABSTRACT. Let \( n \geq 1 \) be an integer and \( S \) be the set of Unitary Divisor of \( n \). Then the set \( S^* = \{ s, n - s/s \in S \text{ and } n \neq s \} \) is a symmetric subset of the group \((\mathbb{Z}_n, \oplus)\), the additive Abelian group of integers modulo \( n \). The Cayley graph of \((\mathbb{Z}_n, \oplus)\) associated with the above symmetric subset \( S^* \) is called the Unitary Divisor Addition Cayley graph and it is denoted by \( G_n(D) \). That is the graph \( G_n(D) \) is the graph whose vertex set is \( V = \{0, 1, \ldots, (n - 1)\} \) and the edge set \( E \) is the set of all ordered pairs of vertices \( x, y \) such that \( x + y \in S^* \). In this paper, we discuss the degree of the vertices and the total number of edges and some properties of Unitary Divisor Addition Cayley graph \( G_n(D) \).

1. INTRODUCTION

A graph \( G \) is a pair \((V, E)\), where \( V = \{0, 1, \ldots, (n - 1)\} \), be the vertex set and \( E \) is a set of unordered pairs of elements of \( V \) are the edges of \( G \). The degree of a vertex \( v \), \( d(v) \) in \( G \) is the number of edges incident at \( v \). If the degree of each vertex is equal, say \( r \) in \( G \), then \( G \) is called \( r \)-regular graph. A graph is called \((r_1, r_2)\)-semi regular if its vertex set can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that all the vertices in \( V_i \) are of degree for \( i = 1, 2 \). Vertices \( u \) and \( v \) of a graph \( G \) are adjacent if \( uv \in E(G) \). Throughout the text, we consider non-trivial, finite, undirected graph with no loops or multiple edges. For standard terminology and notation in graph theory we follow [1],[4].

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For a positive integer \( n > 1 \), the Unitary Addition Cayley graph \( G_n \) is the graph whose vertex set is \( \mathbb{Z}_n \), the integers modulo \( n \) and if \( U_n \) denotes set of all units of the ring \( \mathbb{Z}_n \), then two vertices \( a, b \) are adjacent if and only if \( a + b \in U_n \) [2] and also refer [5], [6].

Let \( n \geq 1 \) be an integer and \( S \) be the set of unitary divisor of \( n \). Then the set \( S^* = \{ s, n - s/s \in S, n \neq s \} \) is a symmetric subset of the group \((\mathbb{Z}_n, \oplus)\), the additive abelian group of integers modulo \( n \). The Cayley graph of \((\mathbb{Z}_n, U_n)\), associated with the above symmetric subset \( S^* \) is called the Unitary Divisor Cayley graph \( G(\mathbb{Z}_n, U_n) \). That is \( G(\mathbb{Z}_n, U_n) \) is the graph whose vertex set \( V = \{0, 1, \ldots, (n - 1)\} \) and the edge set \( E \) is the set of all ordered pairs of vertices \( x, y \), such that \( x - y \in S^* \) or \( y - x \in S^* \), see [3].

Now motivated by the Unitary Addition Cayley graph and Unitary Divisor Cayley graph we introduce the Unitary Divisor Addition Cayley graph as follows: Let \( n \geq 1 \) be an integer and \( S \) be the set of Unitary Divisor of \( n \). Then the set \( S^* = \{ s, n - s/s \in S \text{ and } n \neq s \} \) is a symmetric subset of the group \((\mathbb{Z}_n, \oplus)\), the additive abelian group of integers modulo \( n \) the Cayley graph of \((\mathbb{Z}_n, \oplus)\) associated with the above symmetric subset \( S^* \) is called the Unitary Divisor Addition Cayley graph and it is denoted by \( G_n(D) \). That is the graph \( G_n(D) \) is the graph whose vertex set is \( V(G_n(D)) = \{0, 1, \ldots, (n - 1)\} \) and the edge set \( E \) is the set of all ordered pairs of vertices \( x, y \), such that \( x + y \in S^* \).

The diameter of graph is the maximum distance between the pair of vertices. The number of edges in the maximum matching of \( G \) is called the matching number. Let \( G(V, E) \) be a connected with vertex set \( V \) and edge set \( E \). A subset of \( V \) is called independence if its vertices are mutually non-adjacent. A graph is Eulerian if the graph both connected and has a closed trial(a walk with no repeated edges) containing all edges of the graph. A connected graph has an Euler cycle if and only if every vertex has even degree. A Hamiltonian path is an undirected or directed graph that visits each vertex exactly once.
2. Some Examples of Unitary Divisor Addition Cayley Graph

\[\text{Figure: } G_2(D)\]
\[\text{Figure: } G_3(D)\]
\[\text{Figure: } G_4(D)\]
\[\text{Figure: } G_5(D)\]

Theorem 2.1. Let \(m\) be any vertex of Unitary Divisor Addition Cayley graph \(G_n(D)\), then the degree of \(m\) is
\[
d(m) = \begin{cases} 
\lvert S^* \rvert & \text{if } 2m \pmod{n} \notin S^* \\
\lvert S^* \rvert - 1 & \text{if } 2m \pmod{n} \in S^*.
\end{cases}
\]

Proof. Consider Unitary Divisor Addition Cayley graph \(G_n(D)\) with vertex set \(V(G_n(D)) = \{0, 1, \ldots, (n-1)\}\). Let \(m\) be any vertex in \(V(G_n(D))\). Let \(S\) be the set of unitary divisor of \(n\) and \(S^* = \{s, n-s/s \in S \text{ and } n \neq s\}\). The graph \(G_n(D)\) is \(|S^*|\) (equal to 2)-regular if \(n = 2^\gamma, \gamma > 1\), and \(|S^*|, |S^*| - 1\)-semi regular otherwise.

The vertex \(0 \in V(G_n(D))\) and \(0 \notin S^*\), 0 is adjacent to all the vertices in \(S^*\). Hence degree of 0 is \(|S^*|\). That is \(d(0) = |S^*|\). If the vertex \(m \in V(G_n(D))\) then either \(m \in S^*\) or \(m \notin S^*\).

Suppose \(m \in S^*\) and \(m = n - m\). If \(m = n - m\) implies \(2m = n\). Take \((mod n)\) on both sides we get \(2m \pmod{n} = n \pmod{n}\). That is \(2m \pmod{n} = 0 \notin S^*\). The \(m\) vertex is adjacent to all the vertices in \(S^*\). Hence degree of \(m\) is \(|S^*|\). That is \(d(m) = |S^*|\).

Suppose \(m \in S^*\) and \(m \neq n - m\). Also \(m \notin S^*\). Then \(2m \pmod{n} \neq n \pmod{n}\). Let \(2m \pmod{n} = k \pmod{n}\). If \(k \notin S^*\) then the vertex \(m\) is adjacent to all the vertices in \(S^*\). Hence degree of \(m\) is \(|S^*|\). That is \(d(m) = |S^*|\). If \(k \in S^*\) then \(d(m) = |S^*| - 1\).

Hence \(d(m) = \begin{cases} 
\lvert S^* \rvert & \text{if } 2m \pmod{n} = k \notin S^* \\
\lvert S^* \rvert - 1 & \text{if } 2m \pmod{n} = k \in S^*.
\end{cases}\)

\(\square\)

Remark 2.1. If \(n = 2^\gamma, \gamma > 1\), then the Unitary Divisor is 1. Hence \(S^* = \{1, n-1\}\). The Unitary Divisor Addition Cayley graph is a cycle. Thus \(d(n) = 2\).
Remark 2.2. If $n = p^\alpha$, $\alpha \geq 1$, then $S^* = \{1, n - 1\}$. The Unitary Divisor Addition Cayley graph is a path. The vertices $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ have degree 1 and all other vertices have degree 2.

Theorem 2.2. The total number of edges in the Unitary Divisor Addition Cayley graph $G_n(D)$ is

$$
\begin{cases}
n & \text{if } n = 2^\gamma, \gamma > 1 \\
\frac{|S^*|(n-1)}{2} & \text{if } n \text{ is odd} \\
\frac{|S^*|(n-1)+1}{2} & \text{if } n = 2p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r} \\
\frac{|S^*|(n-1)}{2} + 1 & \text{if } n = 2p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r}.
\end{cases}
$$

Proof. Suppose that $n = 2^\gamma, \gamma > 1$.

Then 1 is the only unitary divisor of $n$, so that $S^* = \{1, n - 1\}$. The Unitary Divisor Addition Cayley graph is a cycle. So each vertex of degree is 2. Therefore number of edges is $n$.

Suppose $n$ is odd, then $n - |S^*|$ vertices having degree $|S^*|$ and $|S^*|$ vertices having degree $|S^*| - 1$. Therefore number of edges in $G_n(D)$ is $\frac{(n-|S^*|)|S^*|+|S^*|(|S^*| - 1)}{2} = \frac{|S^*|(n-1)}{2}$.

Suppose $n = 2p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r}$. Here $(n - |S^*| + 1)$ vertices having degree $|S^*|$ and $|S^*| - 1$ vertices having degree $|S^*| - 1$. Therefore number of edges is $\frac{(n-|S^*|+1)|S^*|+|S^*|(|S^*| - 1)}{2} = \frac{|S^*|(n-1)+1}{2}$.

Suppose $n = 2p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r}$.

Here $(n - |S^*| + 2)$ vertices having degree $|S^*|$ and $|S^*| - 2$ vertices having degree $|S^*| - 1$. The number of edges is $\frac{(n-|S^*|+2)|S^*|+(|S^*| - 2)(|S^*| - 1)}{2} = \frac{|S^*|(n-1)}{2} + 1$. □

Theorem 2.3. The Unitary Divisor Addition Cayley graph is Eulerian iff $n = 2^\gamma$, $\gamma > 1$.

Proof. Suppose $G_n(D)$ is Eulerian. If possible $n$ is other than $2^\gamma$, $\gamma > 1$.

A graph is Eulerian iff $G$ is connected and its vertices all have even degree. Therefore $G_n(D)$ is not Eulerian, a contradiction to our assumption.

Next suppose $n = 2^\gamma$ and $\gamma > 1$, then the degree of each vertex is $|S^*|$ and $|S^*| = 2$ is even. Hence the result. □

Theorem 2.4. The Unitary Divisor Addition Cayley graph is Hamiltonian for $n$ is even and also connected for all $n$.

Proof. Now we construct a cycle $C = (0, n - 1, 2, n - 3, 4, \ldots, n - 4, 3, n - 2, 1, 0)$. 

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Since the cycle \(C\) contains all the vertices of \(G_n(D)\) exactly once, \(C\) is a Hamiltonian cycle of \(G_n(D)\). Thus \(G_n(D)\) is Hamiltonian.

Suppose \(n\) is even, the Unitary Divisor Addition Cayley graph is Hamiltonian. Thus, find a Hamiltonian path \(P = (n - 1, 2, n - 3, 4, \ldots, n - 4, 3, n - 2, 0)\). Then \(G_n(D)\) is connected.

Suppose \(n\) is odd, there exists a path \((n + 1, 2, \ldots, 3, n - 2, 1, 0, n - 1, 2, n - 3, \ldots, n - 1)\). Thus \(G_n(D)\) is connected.

Theorem 2.5. The Unitary Divisor Addition Cayley graph is bipartite iff \(n = 2^\alpha, \alpha > 1\) and \(n = p^m, m \geq 1; p\) is prime.

Proof. Suppose \(n = 2^\alpha, \alpha > 1\) and \(n = p^m, m \geq 1\). We can split the vertex set into two parts \(X = \{0, 2, 4, \ldots, n - 2\}\) and \(Y = \{1, 3, \ldots, n - 1\}\) and \(V = X \cup Y\). Hence \(G_n(D)\) is bipartite.

Theorem 2.6. The diameter of Unitary Divisor Addition Cayley graph is

\[
\begin{cases}
n - 1 & \text{if } n = p^m \\
\frac{n}{2} & \text{if } n = 2^\alpha \\
2 & \text{if } n = 6, 12 \\
3 & \text{otherwise}.
\end{cases}
\]

Proof. Suppose \(n = p^m, m \geq 1\); it is a bipartite graph with \(X = \{1, 3, \ldots, n - 1\}\) and \(Y = \{0, 2, \ldots, n - 2\}\). It is also a path graph. Hence diameter=\(n - 1\).

Suppose \(n = 2^\alpha, \alpha > 1\); the Unitary Divisor Addition Cayley graph is bipartite, the vertex set can split into two parts \(X = \{0, 2, 4, \ldots, n - 2\}\) and \(Y = \{1, 3, 5, \ldots, n - 1\}\), it is a cycle. Hence diameter is \(\frac{n}{2}\).

Suppose \(n = 6\) and \(n = 12\), there exists a path \((1, 0, n - 1)\) of length 2. Hence diameter=2.

Suppose \(n\) is odd. Then there exists two non adjacent vertices \(x\) and \(y\) in vertex set in \(G_n(D)\) such that \(x\) is even and \(y\) is odd and have no common neighbour. Let \(z\) be a neighbour of \(x\) in \(S^*\) and odd, there have a common neighbour \(u\). Hence we take a path \(x - z - u - y\) of length 3. Hence diameter =3.

Theorem 2.7. The matching number of Unitary Divisor Addition Cayley graph is

\[
\begin{cases}
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n - 1}{2} & \text{if } n \text{ is odd}.
\end{cases}
\]
Proof. In $G_n(D)$, the generating set $S^*$ must contain 1.

Suppose $n$ is even. The edge set $E_1 = \{(0, 1), (n - 1, 2), (n - 2, 3), \ldots, \left(\frac{n+2}{2}, \frac{n}{2}\right)\}$ is an independent set in $G_n(D)$ and $|E_1| = \frac{n}{2}$. So the matching number is $\frac{n}{2}$.

Suppose $n$ is odd. The edge set $E_2 = \{(0, 1), (n - 1, 2), \ldots, \left(\frac{n+3}{2}, \frac{n-1}{2}\right)\}$ is an independent set in $G_n(D)$, and $|E_2| = \frac{n-1}{2}$. Therefore matching number is $\frac{n-1}{2}$.

\[ \square \]

REFERENCES


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