ON THE WIENER INDEX OF $F_H$ SUMS OF GRAPHS

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ABSTRACT. Wiener index is the first among the long list of topological indices which was used to correlate structural and chemical properties of molecular graphs. In [5] M. Eliasi, B. Taeri defined four new sums of graphs based on the subdivision of edges with regard to the cartesian product and computed their Wiener index. In this paper, we define a new class of sums called $F_H$ sums and compute the Wiener index of the resulting graph in terms of the Wiener indices of the component graphs so that the results in [5] becomes a particular case of the Wiener index of $F_H$ sums for $H = K_1$, the complete graph on a single vertex.

1. INTRODUCTION

A simple graph $G$ is connected if every pair of vertices are connected by a path. The distance $d(u, v)$ between any two vertices in a connected graph is the length (number of edges) of the shortest path between them. The concept of distance in graph is of vital importance as it is the basic tool to study the topological aspects of graphs, one among them is the Wiener index named after H. Wiener [11]. Wiener index of a graph $G$ denoted by $W(G)$ is defined as the sum of the distance between all pairs of vertices on a connected graph.

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v).$$

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Key words and phrases. Wiener index ($W(G)$), $F_H$ sums.
The origin of Wiener index itself comes from a study on relationship between boiling points and structural aspects of paraffin molecules [11]. Later in molecular graph theory, Wiener index found its supreme importance in studying structural as well as physical composition of various chemical graphs [1]. In [3] H Hosoya introduced a polynomial associated with the Wiener index called Wiener polynomial and obtained the Wiener index as the derivative of this polynomial at unity. Later this polynomial was renamed as Hosoya polynomial. For further results about Wiener index and Hosoya polynomial see [1,10].

Computing the topological indices of various graph operations has been a subject of recent research. Y. N Leh, I. Gutman computed the Wiener index of various graph operations such as product, join, composition [13]. D Stevanović computed the Wiener polynomial of product, join, composition by generalising the earlier results [10]. Cvetković proposed four new graphs $S, R, Q, T$ based on subdivision of edges in $G$. In [12] W Yan et al computed the Wiener index of each subdivision graphs $S, R, Q, T$ in terms of the parent graph as well as the other subdivision related graphs. In [5] M. Eliasi, B. Taeri defined a new operation on graphs called $F$ sums based on these four subdivision graphs and computed the Wiener index of $F$ sum in terms of the Wiener index of the component graphs. H Deng et al computed the Zagreb indices of $F$ sums $G$ and S. Akhter, M Imran computed the forgotten index of $F$ sums [9]. The generalised version of $F$ sums called generalized $F_k$ sums was introduced by J.B Liu et al and they computed the Zagreb indices of the sums in terms of its factor graphs [7].

Based on the subdivisions $S, R, Q, T$ of a graph $G$, four subdivisions with respect to a graph $H (S_H, R_H, Q_H, T_H)$ can be proposed by introducing a new graph $H$ corresponding to each edge of the parent graph and joining the endvertices of each edge to all vertices to the corresponding copy of $H$. The graph $S_H(G)$ is the edge corona of $G$ and $H$ and all other subdivisions are analogous versions of $R, Q, T$. In this paper we obtain the inter relationship of Wiener index of four graphs $S_H, R_H, Q_H$ and $T_H$. We also define a new sum called $F_H$ sums, an analogous version of $F$ sum in terms of $F_H \in \{S_H, R_H, Q_H, T_H\}$ and compute the Wiener index of $F_H$ sums. Thus, the results established in [5] will be particular case for $(H = K_1)$ of the results in this paper.
Let $G_1, G_2$ be two graphs with vertex set $V_1(G), V_2(G)$ and edge set $E_1(G)$ and $E_2(G)$ respectively. Let $H$ be any graph. Then the four graphs associated with $H$ are $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$ and are as defined as follows:

1. $S_H(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a copy of $H$ and making every vertex in the $i$th copy of $H$ adjacent to the end vertices of $e_i$ for each $e_i \in E(G_1)$. That is, $S_H(G_1)$ is a graph with vertex set $V(S_H(G_1)) = V(G_1) \cup V_v(H)$, where $V_v(H) = \bigcup_{i=1}^{E(G_1)} V_i(H)$, $V_i(H) = V(H) \forall i$ and the edge set $E(S_H(G_1)) = \{(v, h), (u, h) : e = vu \in E(G_1), h \in V_v(H) \} \cup E_v(H)$ where $E_v(H) = \bigcup_{i=1}^{E(G_1)} E_i(H)$, $E_i(H) = E(H), \forall i$.

2. $R_H(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a copy of $H$ and making every vertex in the $i$th copy of $H$ adjacent to the end vertices of $e_i$ for each $e_i \in E(G_1)$ also keeping every edge in $G_1$ as well. That is, $R_H(G_1)$ is a graph with vertex set $V(R_H(G_1)) = V(G_1) \cup V_v(H)$ and edge set $E(R_H(G_1)) = \{(v, h), (u, h) : e = vu \in E(G_1), h \in V_v(H) \} \cup E_v(H) \cup E(G_1)$, where $V_v(H) = \bigcup_{i=1}^{E(G_1)} V_i(H)$, $V_i(H) = V(H) \forall i$, $E_v(H) = \bigcup_{i=1}^{E(G_1)} E_i(H)$, $E_i(H) = E(H), \forall i$.

3. $Q_H(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a copy of $H$ and making every vertex in the $i$th copy of $H$ adjacent to the end vertices of $e_i$ for each $e_i \in E(G_1)$ along with edges joining all the vertices in the $i$th copy of $H$ to all the vertices in the $j$th copy of $H$ whenever $e_i$ adjacent to $e_j$ in $G_1$. That is, $Q_H(G_1)$ is a graph with vertex set $V(Q_H(G_1)) = V(G_1) \cup V_v(H)$ and edge set $E(Q_H(G_1)) = \{(v, h), (u, h) : e = vu \in E(G_1), h \in V_v(H) \} \cup E_v(H) \cup E(H, V H_s)$ where $V_v(H) = \bigcup_{i=1}^{E(G_1)} V_i(H)$, $V_i(H) = V(H) \forall i$, $E(H, V H_s) = \{(h_e, h_s) : h_e \in V(H_e), h_s \in V(H_s) \}, E_v(H) = \bigcup_{i=1}^{E(G_1)} E_i(H)$, $E_i(H) = E(H), \forall i$ and $H_e, H_s$ are the copies of $H$ corresponding to the edge $e, s \in E(G_1)$ and $e, s$ are adjacent in $G_1$.

4. $T_H(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a copy of $H$ and making every vertex in the $i$th copy of $H$ adjacent to the end vertices of $e_i$ for each $e_i \in E(G_1)$ along with edges joining all the vertices in the $i$th copy of $H$ to all the vertices in the $j$th copy of $H$.
whenever $e_i$ adjacent to $e_j$ in $G_1$ and keeping every edge of $G_1$ as well. That is, $T_H(G_1)$ is a graph with vertex set $V(T_H(G_1)) = V(G_1) \cup V_i(H)$ and edge set $E(T_H(G_1)) = E(G_1) \cup \{(v, h), (u, h) : e = vu \in E(G_1), h \in V(H)\} \cup E_i(H) \cup E(G_1) \cup E(H_e V H_s)$ where $V_i(H) = \bigcup_{i=1}^{E(G_1)} V_i(H)$, $V_i(H) = V(H) \forall i$, $E(H_e V H_s) = \{(h_e, h_s) : h_e \in V(H_e), h_s \in V(H_s)\}$, $E_i(H) = \bigcup_{i=1}^{E(G_1)} E_i(H)$, $E_i(H) = E(H) \forall i$ and $H_e, H_s$ are the copies of $H$ corresponding to the edge $e, s \in E(G_1)$ and $e, s$ are adjacent in $G_1$. $T_H(G_1)$ is called the total graph associated with $H$.

Corresponding to the four new graphs $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$, we define four new sums called $F_H$ sums associated with the graph $H$. Let $F_H$ be any one of the symbols $S_H, R_H, Q_H, T_H$. The $F_H$ sum of $G_1$ and $G_2$ is denoted by $G_1 +_{F_H} G_2$, is a graph with vertex set $V(G_1 +_{F_H} G_2) = V(F_H(G_1)) \times V(G_2)$ and the edge set $E(G_1 +_{F_H} G_2) = \{(a, b)(c, d) : a = c \in V(G_1) and bd \in E(G_2) or ac \in E(F_H(G_1)) and b = d \in V(G_2)\}$. We consider the newly added vertices as white vertices and already existing vertices as black vertices. Let

![Graphs](image)
Then in, $V(H) = \{u_1, u_2, \ldots, u_p\}$ in this discussion. By $e_{ij}$ we mean the vertex $u_j$ in the $i$th copy of $H$ corresponding to the edge $e_i \in E(G_1)$. Figure 1 is an illustration with $G_1 = P_4, G_2 = P_3$ and $H = P_2$

3. Distance in $F_H(G), F_H(G) = \{S_H(G), R_H(G), Q_H(G), T_H(G)\}$

Let $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $H$ be any graph with $V(H) = \{u_1, u_2, \ldots, u_p\}$. In each $S_H(G), R_H(G), Q_H(G), T_H(G)$, the vertex $e_{ij}$ is as defined above. The edge connecting $u, v$ on a path is denoted by $u \rightarrow v$.

**Lemma 3.1.** Let $G$ be a connected graph and $H$ be any graph, $u, v \in V(G)$. Then

$$d_G(u, v) = \frac{ds_H(G)(u, v)}{2} = d_{R_H(G)}(u, v) = d_{T_H(G)}(u, v) = (d_{Q_H(G)}(u, v)) - 1.$$ 

**Proof.** Let the vertices $u, v$ in $G$ are connected by a shortest path of $P : u = v_0 \rightarrow^{e_1} v_1 \rightarrow^{e_2} v_2 \ldots \rightarrow^{e_{t-1}} v_{t-1} \rightarrow^{e_t} v_t = v$ length $t$. Now fix a vertex $u_j \in V(H)$. Then

- $S_H(G)$: Since each edge in $G$ is replaced by a graph $H$ in $S_H(G)$ the edge $v_{i-1} \rightarrow^{e_i} v_i$ in $P$ can be replaced by the shortest path $v_{i-1} \rightarrow e_{ij} \rightarrow v_i$ of length 2. Thus, every edge in $P$ is replaced by a path of length 2 in $S_H(G)$. The new shortest path connecting $u, v$ in $S_H(G)$ is $P_{S_H} : u = v_0 \rightarrow e_{ij} \rightarrow v_1 \rightarrow e_{2j} \rightarrow v_2 \ldots \rightarrow e_{(t-1)j} \rightarrow v_{t-1} \rightarrow e_{tj} v_t = v$

- $R_H(G), T_H(G)$: The path $P$ itself is a $u \rightarrow v$ path in $R_H(G), T_H(G)$. Now we show that $P$ is the shortest path, For let $P^*$ be the shortest $u \rightarrow v$ path in $R_H(G)$, if $P^*$ consist only the vertices $v \in V(G)$, then $P^* = P$. If not, let $e_i$ be a vertex other than $v \in V(G)$, then the edge $u_i \rightarrow e_i \rightarrow v_i$ can be replaced by the edges $u_i v_i$, Thus $P^*$ is not the shortest in $R_H(G)$. Similarly, Let $P^*$ be the shortest path in $T_H(G)$, if $P^*$ consist only the vertices of $G_1$ then $P = P^*$, if not there exist a section of $P^*$ of the form $u_i \rightarrow e_i \rightarrow e_{i+1} \rightarrow \ldots \rightarrow e_j \rightarrow u_j$ of length $l + 1$ with all the internal vertices of this section is in $V(e(H))$. Then, this section of $P^*$ can be replaced by smaller section $u_i \rightarrow u_{i+1} \rightarrow \ldots u_{j-1} \rightarrow u_j$ of length $l$. Thus $P^*$ is not the shortest, so $P$ is the shortest path in $R_H(G), T_H(G)$.

- $Q_H(G)$: In $Q_H(G)$ to go from one vertex $u \in V(G)$ to another vertex $v \in V(G)$ we have to essentially go through an edge in $G$. Also the path...
$e_{ij} \rightarrow v_i \rightarrow e_{(i+1)j}$ can be replaced by a shorter path $e_{ij} \rightarrow e_{(i+1)j}$. Thus a path $P$ of length $l$ in $G$

$$P : v_0 \rightarrow^{e_1} v_1 \rightarrow^{e_2} \ldots \rightarrow^{e_{l-1}} v_{l-1}$$

corresponds to a path of length $l + 1$ in $Q_{H(G_1)}$ of the form

$$P_{Q_{H(G)}} : v_0 \rightarrow e_{1j} \rightarrow e_{2j} \ldots e_{(l-1)j} \rightarrow e_{(l)j}$$

\[ \square \]

**Lemma 3.2.** Let $G$ be a connected graph and $H$ be any graph with $e_{ij}, e_{ik} \in V(H) \cap V(F_{H(G)})$ where $F_H = S_H$ or $R_H$ or $Q_H$ or $T_H$ (vertices belonging to the same component of $H$). Then

$$d_{F_H}(e_{ij}, e_{ik}) = \begin{cases} 
1, & \text{if } (v_j, v_k) \in E(H) \\
2, & \text{otherwise} 
\end{cases}$$

**Proof.** For each path other than the edge $(e_{ij}, e_{ik}) \in E(F_{H(G)})$, we can replace the path by a shorter path $e_{ij} \rightarrow v_i \rightarrow e_{ik}$ of length 2. \[ \square \]

**Lemma 3.3.** Let $G$ be a connected graph and $H$ be any other graph with $e_{ij}, e_{tk} \in V(H) \cap V(F_{H(G)})$ where $F_H = S_H$ or $R_H$ or $Q_H$ or $T_H$ (vertices belonging to the different component of $H$). Then

$$\frac{d_{S_H(G)}(e_{ij}, e_{tk})}{2} = d_{R_H(G)}(e_{ij}, e_{tk}) - 1 = d_{Q_H(G)}(e_{ij}, e_{tk}) = d_{T_H(G)}(e_{ij}, e_{tk})$$

**Proof.** Fix a vertex $u_j \in V(H)$, Let $e, f \in E(G)$ and the shortest path from $e$ to $f$ of length $l$ in the line graph $L(G)$ be

$$e = e_0 \rightarrow^{v_1} e_1 \rightarrow^{v_2} e_3 \ldots \rightarrow^{v_l} e_l = f.$$

Then the corresponding shortest path of length $l$ in $Q_{H(G)}, T_{H(G)}$ is

$$e_{0j} \rightarrow^{v_1} e_{1j} \rightarrow^{v_2} e_{3j} \ldots \rightarrow^{v_l} e_{lj}.$$

From this path, we obtain a shortest path of length $2l$ in $S_{H(G)}$ as

$$e_{0j} \rightarrow v_1 \rightarrow e_{1j} \rightarrow v_2 \rightarrow e_{3j} \ldots \rightarrow v_l \rightarrow e_{lj}.$$

Similarly, from the above path, we obtain a shortest path of length $l + 1$ in $R_{H(G)}$ as

$$e_{0j} \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_l \rightarrow e_{lj}.$$

\[ \square \]
Lemma 3.4. Let $G$ be a connected graph and $H$ be any other graph with $v \in V(G)$, $e_{ij} \in V(H) \cap V(F_H(G))$ where $F_H = S_H$ or $R_H$ or $Q_H$ or $T_H$. Then
\[
\frac{d_{S_H(G)}(v, e_{ij}) + 1}{2} = d_{R_H(G)}(v, e_{ij}) - 1 = d_{Q_H(G)}(v, e_{ij}) = d_{T_H(G)}(v, e_{ij}).
\]

Proof. Consider a shortest path of length $2l - 1$ in $S_H(G)$ as
\[e_{0j} \rightarrow v_1 \rightarrow e_{1j} \rightarrow v_2 \rightarrow e_{3j} \ldots \rightarrow v_l.\]

From this, we obtain the shortest path of length $l$ in $R_H(G)$ as
\[e_{0j} \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_l.\]

The corresponding shortest path of length $l$ in $Q_H(G)$ is
\[e_{0j} \rightarrow v_1 e_{1j} \rightarrow v_2 e_{3j} \ldots \rightarrow v_l.\]

We choose either one among the above two paths as the shortest path of length $l$ in $T_H(G)$. □

From this lemmas, we obtain the relationship of wiener index among the four graphs as

Theorem 3.1. Let $G$ be a connected graph and $H$ be any graph with vertex sets $V(G)$, $V(H)$ and edge set $E(G)$, $E(H)$ respectively. Then

1. $W(S_H(G)) = 2W(T_H(G)) - |V(G)||E(G)||V(H)|$,
2. $W(S_H(G)) = 2W(Q_H(G)) - |V(G)|||V(G)| - 1| - |V(G)||E(G)||V(H)|$,
3. $W(S_H(G)) = 2W(R_H(G)) - 2(|E(G)| - 1)|V(H)|^2 - |V(G)||E(G)||V(H)|$.

Proof. We divide the sum into three parts as
\[
\sum_{u,v \in V(F_H(G))} d_{F_H(G)}(u, v) = \sum_{u,v \in V(G)} d_{F_H(G)}(u, v) + \sum_{e_{ij}, e_{kt} \in V(F_H(G)) \cap V(H)} d_{F_H(G)}(e_{ij}, e_{kt})
\]
\[+ \sum_{u \in V(G) e_{ij} \in V(F_H(G)) \cap V(H)} d_{F_H(G)}(u, e_{ij})
\]

and by using Lemmas 1-4 we get the desired result. □
4. POLYNOMIALS ASSOCIATED WITH $F_H(G)$

The Wiener index of four graphs $F_H(G) = \{S_H(G), R_H(G), Q_H(G), T_H(G)\}$ are mutually related. A similar observation can be done in the case of Hosoya (Wiener) polynomials as well. Throughout this section we denote $W(G; p)$ as the Hosoya (Wiener) polynomial of $G$ in the variable $p$.

**Theorem 4.1.** Let $G$ be a connected graph and $H$ be any graph. Define

$$A = \{e \in E(T_H(G)) : e = (u, v), u \text{ or } v \in V(G)\}.$$ 

Then

$$W(S_H(G); p) = \frac{W(T_H(G); p^2)}{p} + \left(1 - \frac{1}{p}\right) W(G; p^2) + \left(1 - \frac{1}{p}\right) W(T_H(G)/A; p^2).$$

**Proof.** Splitting the sum into three different parts,

$$W(S_H(G); p) = \sum_{u,v \in V(G)} p^{d_{SH}(G)}(u,v) + \sum_{e_{ij},e_{kl} \in V(F_H(G) \cap V(H))} p^{d_{SH}(G)}(e_{ij},e_{kl})$$

$$+ \sum_{u \in V(G) | e_{ij} \in V(F_H(G) \cap V(H))} p^{d_{SH}(G)}(e_{ij})$$

$$= \sum_{u,v \in V(G)} p^{2d_{TH}(G)}(u,v) - 1 + \sum_{e_{ij},e_{kl} \in V(F_H(G) \cap V(H))} p^{2d_{TH}(G)}(e_{ij},e_{kl}) - 1$$

$$+ \sum_{u \in V(G) | e_{ij} \in V(F_H(G) \cap V(H))} p^{2d_{TH}(G)}(u,v) \left(1 - \frac{1}{p}\right)$$

$$+ \sum_{e_{ij},e_{kl} \in V(S_H(G) \cap V(H))} p^{2d_{TH}(G)}(e_{ij},e_{kl}) \left(1 - \frac{1}{p}\right)$$

$$W(S_H(G), p) = \frac{W(T_H(G); p^2)}{p} + \left(1 - \frac{1}{p}\right) W(G; p^2) + \left(1 - \frac{1}{p}\right) W(T_H(G)/A; p^2).$$

$\square$

**Theorem 4.2.** Let $G$ be a connected graph and $H$ be any graph. Define

$$A = \{e \in E(T_H(G)) : e = (u, v), u \text{ or } v \in V(G)\}.$$ 

Then

$$W(R_H(G), p) = W(T_H(G); p) + (p - 1) W(T_H(G)/A; p).$$
\textbf{Proof.} Splitting the sum into three different parts,

\[ W(R_H(G), p) = \sum_{u,v \in V(G)} p^{d_{R_H(G)}(u,v)} + \sum_{e_{ij}, e_{kl} \in V(R_H(G)) \cap V(H)} p^{d_{R_H(G)}(e_{ij}, e_{kl})} \]

\[ + \sum_{u \in V(G), e_{ij} \in V(R_H(G)) \cap V(H)} p^{d_{R_H(G)}(u, e_{ij})} \]

\[ = \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)} + \sum_{e_{ij}, e_{kl} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kl})} \]

\[ + \sum_{u \in V(G), e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} + (p - 1) \sum_{e_{ij}, e_{kl} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kl})} \]

\[ = W(T_H(G); p) + (p - 1) W(T_H(G)/A; p). \]

\[ \square \]

\textbf{Theorem 4.3.} Let \( G \) be a connected graph and \( H \) be any graph. Then

\[ W(Q_H(G), p) = W(T_H(G); p) + (p - 1) W(G; p). \]

\textbf{Proof.} Splitting the sum into three different parts,

\[ W(Q_H(G), p) = \sum_{u,v \in V(G)} p^{d_{Q_H(G)}(u,v)} + \sum_{e_{ij}, e_{kl} \in V(Q_H(G)) \cap V(H)} p^{d_{Q_H(G)}(e_{ij}, e_{kl})} \]

\[ + \sum_{u \in V(G), e_{ij} \in V(Q_H(G)) \cap V(H)} p^{d_{Q_H(G)}(u, e_{ij})} \]

\[ = \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)+1} + \sum_{e_{ij}, e_{kl} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kl})} \]

\[ + \sum_{u \in V(G), e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} \]

\[ = \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)} + \sum_{e_{ij}, e_{kl} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kl})} \]

\[ + \sum_{u \in V(G), e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} + (p - 1) \sum_{u,v \in V(G)} p^{d_{(G)}(u,v)} \]

\[ = W(T_H(G); p) + (p - 1) W(G; p). \]

\[ \square \]
5. The Wiener Index of $F_H$ Sums

In this section, we compute the Wiener index of the four $F_H$ sums of graphs. The Wiener index of the $F_H$ sums can be computed by finding the distance relations among all kinds of vertices in the sum. So we first obtain the distances between various kinds vertices in $F_H = \{S_H, R_H, Q_H, T_H\}$.

**Lemma 5.1.**

a. Let $G_1, G_2$ be two connected graphs and $H$ be any graph $u = (u_1, u_2)$ be any black vertex. Then for all $v = (v_1, v_2) \in V(G_1 + F_H G_2)$ with $F_H = S_H, R_H$ we have

$$d(u, v|G_1 + F_H G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).$$

b. Let $G_1, G_2$ be two connected graphs and $H$ be any graph. If $e_i, e_j \in E(G_1)$ then for all $u = (e_{ik}, u_2), v = (e_{jt}, v_2) \in V(G_1 + F_H G_2)$ with $u_2 \neq v_2$ $F_H = S_H, R_H$. Then:

$$d(u, v|G_1 + F_H G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } i = j \\ d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2) & \text{if } i \neq j \end{cases}.$$

c. Let $G_1, G_2$ be two connected graphs and $H$ be any graph. If $e_i, e_j \in E(G_1)$ then for all $u = (e_{ik}, u_2), v = (e_{jt}, v_2) \in V(G_1 + F_H G_2)$ with $u_2 = v_2$ $F_H = S_H, R_H$. (White vertices in the same copy). Then:

$$d(u, v|G_1 + F_H G_2) = d(e_{ik}, e_{jt}|F_H(G_1)).$$

**Proof.**

a. Let $x = d(u, v|G_1 + F_H G_2)$, $x_1 = d(u_1, v_1|F_H(G_1))$ and $x_2 = d(u_2, v_2|G_2)$, and let

$$P_1: u_1 = t_0^1 \rightarrow t_1^1 \rightarrow \ldots \rightarrow t_{x_1}^1 \rightarrow v_1,$$

$$P_2: u_2 = s_0^2 \rightarrow s_1^2 \rightarrow \ldots \rightarrow s_{x_2}^2 \rightarrow v_2,$$

be the corresponding shortest paths. Then using these two paths $P_1, P_2$ we easily construct a new path from $u$ to $v$ in $G_1 + F_H G_2$ as

$$P_3: (u_1, u_2) = (t_0^1, s_0^2) \rightarrow (t_1^1, s_0^2) \rightarrow \ldots \rightarrow (t_{x_1-1}^1, s_0^2) \rightarrow (t_{x_1}^1, s_0^2) = (v_1, s_0^2) \rightarrow (v_1, s_1^2) \rightarrow \ldots \rightarrow (v_1, s_{x_2}^2) = (v_1, v_2).$$
Thus, $x \leq x_1 + x_2$. Also corresponding to every path $P$ from $u$ to $v$ in $G_1 + F_H G_2$,

$$P : (u_1, u_2) = (t_0^1, s_0^2) \rightarrow (t_1^1, s_1^2) \rightarrow \ldots \rightarrow (t_{x-1}^1, s_{x-1}^2) \rightarrow (t_x^1, s_x^2) = (v_1, v_2),$$

we construct a path from $u_1$ to $v_1$ as $u_1 = t_0^1 \rightarrow t_1^1 \rightarrow \ldots \rightarrow t_{y_1-1}^1 \rightarrow t_{y_1}^1 = v_1$ in $F_H(G_1)$ and a path of the form $u_2 = s_0^2 \rightarrow s_1^2 \rightarrow \ldots \rightarrow s_{y_2-1}^2 \rightarrow s_{y_2}^2 = v_2$ from $u_2$ to $v_2$ in $G_2$ by replacing every consecutive similar vertex $(uuuu..u)$ by a single vertex $(u)$. Thus $x_1 + x_2 \leq x = y_1 + y_2$, that is

$$d(u, v|G_1 + F_H G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).$$

b. Consider the case $u = (e_{ik}, u_2), v = (e_{jt}, v_2)$ with $u_2 \neq v_2$.

**Case I** $i = j$: Let $e_i = u_i v_i$ and $u_2 = s_0^2 \rightarrow s_1^2 \rightarrow \ldots \rightarrow s_{y_2-1}^2 \rightarrow s_{y_2}^2 = v_2$ be the shortest path of length $y_2$ from $u_2$ to $v_2$ in $G_2$. Fix $u_i$, then we easily construct a shortest path from $u$ to $v$ using the edges $e_{ik}u_i$ and $e_{jt}u_i$ and the path from $u_2$ to $v_2$ as

$$P : (e_{ik}, u_2) = (e_{ik}, s_0^2) \rightarrow (u_i, s_0^2) \rightarrow (u_i, s_1^2) \rightarrow \ldots \rightarrow (u_i, s_{y_2}^2) \rightarrow (e_{jt}, v_2)$$

of length $2 + d(u_2, v_2|G_2)$.

**Case II** $i \neq j$: Let $d(e_{ik}, e_{jt}|F_H(G_1)) = y_1, x = d(u, v|G_1 + F_H G_2), d(u_2, v_2) = x_1$ and let $e_i = p_0^1 \rightarrow p_1^1 \rightarrow \ldots \rightarrow p_{y_1-1}^1 \rightarrow p_{y_1}^1 = e_{jt}$ (where $p_{y_1}^1 \in V(G_1)$) since $F_H = S_H, R_H$ and $u_2 = s_0^2 \rightarrow s_1^2 \rightarrow \ldots \rightarrow s_{x_1-1}^2 \rightarrow s_{x_1}^2 = v_2$ be the corresponding paths. Now we construct the following two paths in $G_1 + F_H G_2$.

$$P_1 : u = (p_0^1, u_2) \rightarrow (p_1^1, u_2) \rightarrow \ldots \rightarrow (p_{y_1-1}^1, u_2) = (p_{y_1-1}^1, s_0^2)$$

$$(p_{y_1-1}^1, s_0^2) \rightarrow (p_{y_1-1}^1, s_1^2) \rightarrow \ldots \rightarrow (p_{y_1-1}^1, s_{x_1}^2) \rightarrow (e_{jt}, v_2) = v$$

of lengths $x_1 + y_1$, thus $x = d(u, v|G_1 + F_H G_2) \leq x_1 + y_1$, to prove the reverse part, we assume that there exist a path $P$ from $u$ to $v$ in $G_1 + F_H G_2$ and proceeding as in the proof of (a) we will establish that $x_1 + y_1 \leq x$. Thus, $d(u, v|G_1 + F_H G_2) = d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2)$.

c. As in the proof of (a) consider the shortest path form $e_{ik}$ to $e_{jt}$ of length $y_1$, $e_{ik} = p_0^1 \rightarrow p_1^1 \rightarrow \ldots \rightarrow p_{y_1-1}^1 \rightarrow p_{y_1}^1 = e_{jt}$ which corresponds to a path $(e_{ik}, u_2) = (p_0^1, u_2) \rightarrow (p_1^1, u_2) \rightarrow \ldots \rightarrow (p_{y_1-1}^1, u_2)$ of
same length from \( u \) to \( v \). Conversely, since \( u_2 = v_2, d(u_2, v_2) = 0 \) every such path \( P \) in \( G_1 + F_H G_2 \) must have same second component thus the shortest path from \( u \) to \( v \) must be of the same length as the shortest path from \( e_{ik} \) to \( e_{jt} \).

\[ \square \]

**Lemma 5.2.**

a. Let \( G_1, G_2 \) be two connected graphs \( H \) be any graph and \( u = (u_1, u_2) \) be any \( u \) black vertex. Then for all \( v = (v_1, v_2) \in V(G_1 + F_H G_2) \) with \( F_H = Q_H, T_H \) we have

\[
d(u, v|G_1 + F_H G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).
\]

b. Let \( G_1, G_2 \) be two connected graphs and \( H \) be any graph. If \( e_i, e_j \in E(G_1) \), then for all \( u = (e_{ik}, u_2), v = (e_{jt}, v_2) \in V(G_1 + F_H G_2) \) with \( u_2 \neq v_2 \) \( F_H = Q_H, T_H \). Then:

\[
d(u, v|G_1 + F_H G_2) =
\begin{cases}
2 + d(u_2, v_2|G_2) & \text{if } i = j \text{ } u_2 \neq v_2 \\
1 + d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2) & \text{if } i \neq j, u_2 \neq v_2.
\end{cases}
\]

c. Let \( G_1, G_2 \) be two connected graphs and \( H \) be any graph. If \( e_i, e_j \in E(G_1) \), then for all \( u = (e_{ik}, u_2), v = (e_{jt}, v_2) \in V(G_1 + F_H G_2) \) with \( u_2 = v_2 \) \( F_H = Q_H, T_H \), (white vertices in the same copy). Then:

\[
d(u, v|G_1 + F_H G_2) = d(e_{ik}, e_{jt}|F_H(G_1)).
\]

**Proof.**

a. Proceed as in the case (a.) of Lemma 5, we easily obtain the result using similar arguments.

b. Case I, \( i = j, u_2 \neq v_2 \): Then proceed as in the Case I of Lemma 5(b) to get the results.

Case II \( i \neq j, u_2 \neq v_2 \): Let \( d(e_{ik}, e_{jt}|F_H(G_1)) = y_1, d(u_2, v_2) = x_1, x = d(u, v|G_1 + F_H G_2) \) and let \( e_{ik} = p^1_0 \rightarrow p^1_1 \rightarrow \ldots p^1_{y_1-1} \rightarrow p^1_{y_1} = e_{jt} \) and \( u_2 = s^2_0 \rightarrow s^2_1 \rightarrow \ldots s^2_{x_1-1} \rightarrow s^2_{x_1} = v_2 \) be the corresponding paths. Let \( u^1_1 v^1_1 = e_i \) and \( u^1_1 v^1_2 = e_j \), also use the fact that the shortest path connecting \( e_{ik}, e_{jt} \) in \( F_H = Q_H, T_H \) must be a path consisting only of vertices in the copies \( H (V_e(H)) \) as every path of \( e_{ik} \rightarrow v^1_1 \rightarrow e_{jt} \) (where \( v^1_1 \) is common vertex of both the edges) can be replaced by a single
Theorem 5.1. Let \( G_1, G_2 \) be two connected graphs and \( H \) be any graph and \( F_H = S_H \) or \( R_H \). Then
\[
W(G_1 + F_H G_2) = |V(G_2)|^2 W(F_H(G_1)) + (|V(G_1)|^2 + (|E(G_1)||V(H)|)^2 + 2|V(G_1)||E(G_1)||V(H)|) W(G_2) + (|V(G_2)|^2 - |V(G_2)|^2) (|E(G_1)||V(H)|).
\]

Proof. We divide the vertex set of \( G_1 + F_H G_2 \) into two different subsets as
\[
A = \{ u = (u_1, v_1) \in V(G_1 + F_H G_2) : u = (u_1, v_1) \in V(G_1) \times V(G_2) \},
B = \{ u = (u_1, v_1) \in V(G_1 + F_H G_2) : u_1 \in V_e(H), v_1 \in V(G_2) \}.
\]

Using this, we find the Wiener index of \( F_H \) Sums in terms of the Wiener index of its component graphs.

Theorem 5.1. Let \( G_1, G_2 \) be two connected graphs and \( H \) be any graph and \( F_H = S_H \) or \( R_H \). Then
\[
W(G_1 + F_H G_2) = |V(G_2)|^2 W(F_H(G_1)) + (|V(G_1)|^2 + (|E(G_1)||V(H)|)^2 + 2|V(G_1)||E(G_1)||V(H)|) W(G_2) + (|V(G_2)|^2 - |V(G_2)|^2) (|E(G_1)||V(H)|).
\]

Proof. We divide the vertex set of \( G_1 + F_H G_2 \) into two different subsets as
\[
A = \{ u = (u_1, v_1) \in V(G_1 + F_H G_2) : u = (u_1, v_1) \in V(G_1) \times V(G_2) \},
B = \{ u = (u_1, v_1) \in V(G_1 + F_H G_2) : u_1 \in V_e(H), v_1 \in V(G_2) \}.
\]
We divide the sum of the distances between the vertices of $G_1 +_{F_H} G_2$ into different components to calculate the Wiener index:

$$W(G_1 +_{F_H} G_2) = \frac{1}{2} \sum_{u,v \in A} d(u,v|G_1 +_{F_H} G_2) + \frac{1}{2} \sum_{u \in A,v \in B} d(u,v|G_1 +_{F_H} G_2) + \frac{1}{2} \sum_{u,v \in B} d(u,v|G_1 +_{F_H} G_2).$$

Now we find the three sums separately. To find the first sum, we use the fact that $\forall u = (u_1, v_1), v = (u_2, v_2) \in A, d(u,v|G_1 +_{F_H} G_2) = d((u_1, v_1), (u_2, v_2)|G_1 +_{F_H} G_2)$ $d((u_1, v_1), (u_2, v_2)|G_1 +_{F_H} G_2) = d(u_1, u_2|F_H(G_1)) + d(v_1, v_2|G_2)$ and Lemma 5.

$A_1$

$$= \frac{1}{2} \sum_{u,v \in A} d(u,v|G_1 +_{F_H} G_2) = \frac{1}{2} \sum_{u,v \in A} d((u_1, v_1), (u_2, v_2)|G_1 +_{F_H} G_2)$$

$$= \frac{1}{2} \sum_{(u_1, v_1), (u_2, v_2)} d(u_1, u_2|F_H(G_1)) + d(v_1, v_2|G_2)$$

$$= \frac{1}{2} \left( \sum_{u_1,u_2 \in V(G_1)} \sum_{v_1,v_2 \in V(G_2)} d(u_1, u_2|F_H(G_1)) + \sum_{u_1,u_2 \in V(G_1)} \sum_{v_1,v_2 \in V(G_2)} d(v_1, v_2|G_2) \right)$$

$$= \frac{1}{2} |V(G_2)|^2 \sum_{u_1,u_2 \in V(G_1)} d(u_1, u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2).$$

Now consider the case where $u = (u_1, v_1) \in V(G_1) \times V(G_2), v = (e_{jt}, v_2), e_{jt} \in V_e(H), v_2 \in V(G_2)$ (or vice versa). By Lemma 5 we have $d(u,v|G_1 +_{F_H} G_2) = d((u_1, e_{jt}, v_2)|G_1 +_{F_H} G_2) = d(u_1, e_{jt}|F_H(G_1)) + d(v_1, v_2|G_2),$

$$\frac{1}{2} \sum_{u \in A,v \in B} d(u,v|G_1 +_{F_H} G_2)$$

$$= \frac{1}{2} \sum_{u \in A,v \in B} d((u_1, e_{jt}), (v_1, v_2)|G_1 +_{F_H} G_2)$$

$$= \frac{1}{2} \sum_{(u_1, v_1), (e_{jt}, v_2)} (d(u_1, e_{jt}|F_H(G_1)) + d(v_1, v_2|G_2))$$
ON THE WIENER INDEX OF $F_H$ SUMS OF GRAPHS

\[ \begin{align*}
&= \frac{1}{2} \sum_{u_1 \in V(G_1)} \sum_{e_{ij} \in V_e(H)} \sum_{v_1, v_2 \in V(G_2)} d(u_1, e_{ij})|F_H(G_1)) \\
&+ \frac{1}{2} \sum_{u_1 \in V(G_1)} \sum_{e_{ij} \in V_e(H)} \sum_{v_1, v_2 \in V(G_2)} d(v_1, v_2) \\
&= \frac{1}{2}|V(G_2)|^2 \sum_{u_1 \in V(G_1)} \sum_{e_{ij} \in V_e(H)} d(u_1, e_{ij})|F_H(G_1)) \\
&+ |V(G_1)||E(G_1)||V(H)||W(G_2).
\end{align*} \]

By considering the reverse case as well, the total distance is

\[ A_2 = \frac{1}{2} \sum_{u \in A, v \in B} d(u, v|G_1 + F_H G_2) \]

\[ = |V(G_2)|^2 \sum_{u_1 \in V(G_1)} \sum_{e_{ij} \in V_e(H)} d(u_1, e_{ij})|F_H(G_1)) \\
+ 2|V(G_1)||E(G_1)||V(H)||W(G_2). \]

Now consider the case where $u = (e_{ij}, v_1), v = (e_{ij}, v_2), e_{ij}, e_{ij} \in V_e(H), v_1, v_2 \in V(G_2)$. We divide the sum into three different parts with respect to $i = j, v_1 \neq v_2, i \neq j, v_1 \neq v_2$ and $i = j, v_1 = v_2$. Let

\[ S_1 = \frac{1}{2} \sum_{u, v \in B} \{d(u, v|G_1 + F_H G_2) : v_1 \neq v_2, i = j\}, \]

\[ S_2 = \frac{1}{2} \sum_{u, v \in B} \{d(u, v|G_1 + F_H G_2) : v_1 \neq v_2, i \neq j\}, \]

\[ S_3 = \frac{1}{2} \sum_{u, v \in B} \{d(u, v|G_1 + F_H G_2) : v_1 = v_2, i = j\}. \]

Then,

\[ (5.1) \quad A_3 = \frac{1}{2} \sum_{u, v \in B} d(u, v|G_1 + F_H G_2) = S_1 + S_2 + S_3. \]
Now,
\[ S_1 = \frac{1}{2} \sum_{(e_{ik}, v_1)(e_{jk}, v_2) \in B} 2 + d(v_1, v_2|G_2) \]
\[ = \frac{1}{2} \sum_{v_1, v_2 \in V(G_2); v_1 \neq v_2} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i = j} 2 \]
\[ + \frac{1}{2} \sum_{v_1, v_2 \in V(G_2); v_1 \neq v_2} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i = j} d(v_1, v_2|G_2) \]
\[ = (|V(G_2)|^2 - |V(G_2)|) |E(G_1)||V(H)| + (|E(G_1)||V(H)|) W(G_2). \]

Similarly,
\[ S_2 = \frac{1}{2} \sum_{u, v \in B} d(e_{ik}, e_{jk}|F_H(G_1)) + d(v_1, v_2|G_2) \]
\[ = \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i \neq j} d(e_{ik}, e_{jk}|F_H(G_1)) \]
\[ + \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i \neq j} d(v_1, v_2|G_2) \]
\[ = \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i \neq j} d(e_{ik}, e_{jk}|F_H(G_1)) \]
\[ + (|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|) W(G_2). \]

Also,
\[ S_3 = \frac{1}{2} \sum_{u, v \in B} \{d(u, v|F_{H+G_2} : v_1 = v_2, i = j]\}
\[ = \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)} \sum_{e_{ik}, e_{jk} \in V_c(H) \atop i = j} d(e_{ik}, e_{jk}|F_H(G_1)), \]

\[ W(G_1 + F_H G_2) = A_1 + A_2 + A_3 \]
\[ = \frac{1}{2} |V(G_2)|^2 \sum_{u_1, u_2 \in V(G_1)} d(u_1, u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2) \]
\[ + |V(G_2)|^2 \sum_{u_1 \in V(G_1)} \sum_{e_{jk} \in V_c(H)} d(u_1, e_{jk}|F_H(G_1)) + 2|V(G_1)||E(G_1)||V(H)|W(G_2) \]
Theorem 5.2. Let \( G_1, G_2 \) be two connected graphs, \( H \) be any graph and \( F_H = Q_H \) or \( T_H \). Then

\[
W(G_1 + F_H G_2) = |V(G_2)|^2 W(F_H(G_1)) + (|V(G_1)|^2 + (|E(G_1)||V(H)|)^2 + 2|V(G_1)||E(G_1)||V(H)| W(G_2) + \frac{1}{2} ((|E(G_1)||V(H)|)^2 + |E(G_1)||V(H)|) (|V(G_2)|^2 - |V(G_2)|)
\]

Proof. Define \( A, B \) as in Theorem 1. Then

\[
W(G_1 + F_H G_2) = \frac{1}{2} \sum_{u,v \in A} d(u, v\,|G_1 + F_H G_2) + \frac{1}{2} \sum_{u \in A, v \in B} d(u, v\,|G_1 + F_H G_2) + \frac{1}{2} \sum_{u,v \in B} d(u, v\,|G_1 + F_H G_2).
\]

As in Theorem 1 we divide the sum into three different parts and calculate the sum of the distances. Now, the first two parts are as same as in Theorem 1, so its enough to find only the third sum. As in the previous cases we break down the third sum into four different parts

\[
\sum_{u,v \in B} d(u, v\,|G_1 + F_H G_2) = C_1 + C_2 + C_3 + C_4.
\]
Let \( u = (e_{ij}, v_1) \), \( v = (e_{jt}, v_2) \), \( e_{ij}, e_{jt} \in V_e(H) \), \( v_1, v_2 \in V(G_2) \). Then,

\[
C_1 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 \neq v_2, i = j\}
\]

\[
C_2 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 = v_2, i \neq j\}
\]

\[
C_3 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 = v_2, i = j\}
\]

\[
C_4 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 \neq v_2, i \neq j\}
\]

By Lemma 6, we have,

\[
C_1 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 \neq v_2, i = j\}
\]

\[
= \frac{1}{2} \sum_{(e_{ik}, v_1)(e_{jt}, v_2) \in B} 2 + d(v_1, v_2|G_2)
\]

\[
= \frac{1}{2} \sum_{v_1, v_2 \in V(G_2) \atop v_1 \neq v_2} \sum_{e_{ik} \in V_e(H) \atop i = j} 2
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2) \atop v_1 \neq v_2} \sum_{e_{ik} \in V_e(H) \atop i = j} d(v_1, v_2|G_2)
\]

\[
= (|V(G_2)|^2 - |V(G_2)|) |E(G_1)||V(H)| + (|E(G_1)||V(H)|) W(G_2)
\]

Now,

\[
C_2 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 = v_2, i \neq j\}
\]

\[
= \frac{1}{2} \sum_{v_1, v_2 \in V(G_2) \atop v_1 = v_2} \sum_{e_{ik} \in V_e(H) \atop i \neq j} d(e_{ik}, e_{jt}|F_H(G_1)).
\]

Similarly,

\[
C_3 = \frac{1}{2} \sum_{u,v \in B} \{d(u,v|G_1 + F_H G_2) : v_1 = v_2, i = j\}
\]

\[
= \frac{1}{2} \sum_{v_1, v_2 \in V(G_2) \atop v_1 = v_2} \sum_{e_{ik} \in V_e(H) \atop i = j} d(e_{ik}, e_{jt}|F_H(G_1)).
\]
Similarly, by Lemma 6,

\[
C_4 = \frac{1}{2} \sum_{u,v \in B} \{ d(u,v|G_1 + F_H G_2) : v_1 \neq v_2, i \neq j \}
\]

\[
= \frac{1}{2} \sum_{u,v \in B} (1 + d(e_{ik}, e_{jl}|F_H(G_1)) + d(v_1, v_2|G_2))
\]

\[
= \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 \neq v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i \neq j} 1
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 \neq v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i \neq j} d(e_{ik}, e_{jl}|F_H(G_1))
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 \neq v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i \neq j} d(v_1, v_2|G_2)
\]

\[
= \left( (|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)| \right) \left( |V(G_2)|^2 - |V(G_2)| \right)
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 \neq v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i \neq j} d(e_{ik}, e_{jl}|F_H(G_1))
\]

Thus we obtain,

\[
W(G_1 + F_H G_2)
\]

\[
= \frac{1}{2} |V(G_2)|^2 \sum_{u_1, u_2 \in V(G_1)} d(u_1, u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2)
\]

\[
+ |V(G_2)|^2 \sum_{v_1 \in V(G_1)} \sum_{e_{jl} \in V_e(H)} d(v_1, e_{jl}|F_H(G_1)) + 2|V(G_1)||E(G_1)||V(H)|W(G_2)
\]

\[
+ \left( |V(G_2)|^2 - |V(G_2)| \right) |E(G_1)||V(H)| + |E(G_1)||V(H)|) W(G_2)
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 \neq v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i \neq j} d(e_{ik}, e_{jl}|F_H(G_1))
\]

\[
+ \frac{1}{2} \sum_{v_1, v_2 \in V(G_2)v_1 = v_2} \sum_{e_{ik}e_{jl} \in V_e(H) i = j} d(e_{ik}, e_{jl}|F_H(G_1))
\]

\[
+ \frac{1}{2} \left( |E(G_1)||V(H)|^2 - |E(G_1)||V(H)| \right) \left( |V(G_2)|^2 - |V(G_2)| \right)
\]
Illustration 1. If $H = K_1$, we obtain the results in [5].

Illustration 2. If $G_1, G_2, H$ are paths $P_n, P_m, P_r$ with $n, m > 3$ respectively, then the Wiener index is:

a. $W(P_n + s_{P_r} P_m) = \frac{m}{6}(2n^3r^2m + n^2r^2m^2 - 6n^2r^2m + 4n^3rm + 2n^2r^2m - 2n^2m^3 - 6n^2rm + 10n^2m^2 - 2nrm^2 + 2n^3m + n^2r^2 + r^2m^2 - 4nrm + 2n^2r - 2nr^2 + n^2m - 6mr^2 - 8nr + 6mr + 6mn + n^2 + r^2 - 6m + 6r)$;

b. $W(P_n + r_{P_r} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2r^2m^2 - 2n^2m^3 - nr^2m + n^3m - 2nrm^2 + n^2m^2 - n^2r^2 + r^2m^2 - 8nrm - 2nr^2 + 2nr^2 + 5mn - 4nr + 6rm - n^2 - r^2 - 6m + 6r)$;

c. $W(P_n + q_{P_r} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2r^2m^2 - 2n^2m^3 + 2nr^2m - 2nrm^2 + n^3m + n^2m^2 - 4n^2r^2 + r^2m^2 - 11nrm - 2n^2r + 8nr^2 - 3r^2m + 3n^2m + 2nm - nr + 9rm - n^2 - 4r^2 - 6m + 3r)$;

d. $W(P_n + t_{P_r} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2r^2m^2 - 2n^2m^3 + 2nr^2m - 2nrm^2 + n^3m + n^2m^2 - 4n^2r^2 + r^2m^2 - 11nrm - 2n^2r + 8nr^2 - 3r^2m + 5nm - nr + 9rm - n^2 - 4r^2 - 6m + 3r)$.

6. Conclusions

In this paper we have defined $F_H$ sums of graphs and obtained the relationship among distances of these four graphs and their Hosoya polynomial. We also computed the Wiener index of $F_H$ sums of two connected graphs. These sums can be defined in terms of various other products such as strong product and lexicographic product. Other topological indices can also be computed for the $F_H$ sums.
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