ON GENERALIZATIONS OF 2-ABSORBING PRIMARY IDEALS IN SEMIGROUPS

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ABSTRACT. Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ is a set of all ideals of a semigroup. We extend the concept of primary and 2-absorbing ideals in semigroups to the context of $\phi$-2-absorbing primary ideals. We say that a proper ideal $A$ of a semigroup $S$ is a $\phi$-2-absorbing primary ideal if for each $a, b, c \in S$ with $abc \in A - \phi(A)$ implies that $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. The aim of this paper is to investigate the concept of $\phi$-2-absorbing primary ideals in semigroups. Finally, we obtain sufficient conditions of a 2-absorbing primary ideal in order to be rephrased a $\phi$-2-absorbing primary ideal in a semigroup.

1. $\phi$-2-ABSORBING PRIMARY IDEALS

In this section, we introduce the concept of $\phi$-2-absorbing primary ideals in semigroups and give its characterizations corresponding to $\phi$-2-absorbing primary ideals in semigroups.

Let $A$ be a subset of a semigroup $S$. Then, the radical (see [1]) of $A$ is defined as $\sqrt{A} = \{a \in S : a^n \in A \text{ for some positive integer } n\}$.

Definition 1.1. Let $S$ be a semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ be the set of all ideals of $S$. A proper ideal $A$ of $S$ is called a $\phi$-2-absorbing primary ideal if for each $a, b, c \in S$ with $abc \in A - \phi(A)$, then $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$.

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We now present the following example satisfying above definition.

**Example 1.** Let $S = \{a, b, c, d, e\}$ be a semigroup with following multiplication given by

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It is easy to see that $\{a, b, d\}$ is a $\phi$-2-absorbing primary ideal of a semigroup $S$.

**Remark 1.1.** It is easy to see that every 2-absorbing primary ideal of a semigroup $S$ is a $\phi$-2-absorbing primary ideal of $S$.

The following example shows that the converse of Remark 1.1 is not true.

**Example 2.** Let $S = \mathbb{Z}^+$. Consider the proper ideal $P = 30\mathbb{Z}^+$ of the semigroup $S$. Define $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ by $\phi(A) = A$ for every $A \in \mathcal{I}(S)$. It is easy to see that $P$ is a $\phi$-2-absorbing primary ideal of $S$. Notice that $2 \cdot 3 \cdot 5 \in P$, but $2 \cdot 3 \not\in \sqrt{P}$ and $3 \cdot 5 \not\in \sqrt{P}$. Therefore $P$ is not a 2-absorbing primary ideal of $S$.

Let $S$ be a semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. Since $A - \phi(A) = A - (A \cap \phi(A))$ for all $A \in \mathcal{I}(S)$, without loss of generality, we will assume that $\phi(A) \subseteq A$. Throughout this paper, as it is noted earlier, if $\phi$ is a function, then we always assume that $\phi(A) \subseteq A$.

**Theorem 1.1.** Let $A$ be a non empty subset of a commutative semigroup $S$. Then the following properties hold.

1. If $A$ is an ideal of $S$, then $\sqrt{A}$ is an ideal of $S$ containing $A$.
2. $\sqrt{A} = \sqrt{\sqrt{A}}$.
3. For each $\phi$-2-absorbing primary ideal $A$ of $S$ if $\sqrt{\phi(A)} \subseteq \phi\left(\sqrt{A}\right)$, then $\sqrt{A}$ is a $\phi$-2-absorbing primary ideal of $S$.
4. For each element $s$ of $S - \sqrt{A}$ if $A$ is a $\phi$-2-absorbing primary ideal of $S$ such that $\sqrt{\phi(A)} \subseteq \phi\left(\sqrt{A}\right)$, then $(\sqrt{A} : s)$ is a $\phi$-2-absorbing primary ideal of $S$ with $\left(\phi(\sqrt{A}) : s\right) \subseteq \phi\left(\sqrt{A} : s\right)$. 

Proof.
1. Assume that $A$ is an ideal of $S$. It is easy to see that, $A \subseteq \sqrt{A}$. Let $a$ and $s$ be any elements of $S$ such that $a \in \sqrt{A}$. Then we have, $a^n \in A$ for some positive integer $n$, which implies that $(sa)^n = s^na^n \in s^nA \subseteq A$. Therefore, $sa \in \sqrt{A}$ and hence $\sqrt{A}$ is an ideal of $S$ containing $A$.

2. Assume that $A$ is a subset of $S$. Obviously, $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$. On the other hand, let $x \in \sqrt{\sqrt{A}}$. Then we have, $x^n \in \sqrt{A}$ for some positive integer $n$, which means that $x^{nm} \in A$ for some positive integer $m$. Therefore, $x \in \sqrt{A}$ and hence $\sqrt{A} = \sqrt{\sqrt{A}}$.

3. Assume that $A$ is an ideal of $S$. Then by part 1, $\sqrt{A}$ is an ideal of $S$. Let $a, b$ and $c$ be any elements of $S$ such that $abc \in \sqrt{A} - \phi(\sqrt{A})$. Thus we have, $abc \notin \phi(\sqrt{A})$ and $(abc)^n \in A$ for some positive integer $n$. Since $\sqrt{\phi(A)} \subseteq \phi(\sqrt{A})$, we have $(abc)^n \notin \phi(A)$, which implies that $(abc)^n \notin A - \phi(A)$. In fact, since $A$ is a $\phi$-2-absorbing primary ideal of $S$, we have $(ab)^n \in A$ or $(bc)^n \in \sqrt{A}$ or $(ac)^n \in A$. Therefore $ab \in \sqrt{A}$ or $bc \in \sqrt{\sqrt{A}}$ or $ac \in \sqrt{A}$ and hence $\sqrt{\sqrt{A}}$ is a $\phi$-2-absorbing primary ideal of $S$.

4. Let $a, b, c$ and $s$ be any elements of $S$ such that $abc \in (\sqrt{A} : s) - \phi(\sqrt{A} : s)$. Since $(\phi(\sqrt{A}) : s) \subseteq \phi(\sqrt{A} : s)$, we have $ab(cs) \in \sqrt{A} - \phi(\sqrt{A})$. Then by parts 2 and 3, $ab \in \sqrt{A}$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which implies that $ab \in (\sqrt{A} : s)$ or $bc \in \sqrt{(\sqrt{A} : s)}$ or $ac \in \sqrt{(\sqrt{A} : s)}$. Consequently, $(\sqrt{A} : s)$ is a $\phi$-2-absorbing primary ideal of $S$.

In the light of the definition of $\phi$-2-absorbing primary ideal in commutative semigroups, we can obtain the following properties.

**Theorem 1.2.** Let $S$ be a commutative semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. If $A$ is a $\phi$-2-absorbing primary ideal of $S$ such that $\sqrt{A}$ is a primary ideal of $S$, then $(A : s)$ is a $\phi$-2-absorbing primary ideal of $S$ for every $s \in S - \sqrt{A}$ with $(\phi(A) : s) \subseteq \phi(A : s)$.

**Proof.** Let $a, b, c$ and $s$ be any elements of $S$ such that $abc \in (A : s) - \phi(A : s)$. Since $(\phi(A) : s) \subseteq \phi(A : s)$, we have $a(bc)s \in A - \phi(A)$. In fact, since $A$ is a $\phi$-2-absorbing primary ideal of $S$, we have $abc \in A$ or $bc \in \sqrt{A}$ or $as \in \sqrt{A}$.

If $bc \in \sqrt{A}$ or $as \in \sqrt{A}$, then $bc \in \sqrt{(A : s)}$ or $a \in \sqrt{(A : s)}$, since $\sqrt{A}$ is a primary ideal of $S$ and $s \in S - \sqrt{A}$. Next, if $abc \in A$, then $abc \in A - \phi(A)$. Therefore, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. In any case, we have $ab \in (A : s)$ or
\(bc \in \sqrt{(A : s)}\) or \(ac \in \sqrt{(A : s)}\). Consequently, \((A : s)\) is a \(\phi\)-2-absorbing primary ideal of \(S\).

In the following result, we give an equivalent definition of \(\phi\)-2-absorbing primary ideals in a commutative semigroup.

**Theorem 1.3.** Let \(\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}\) be a function. The following conditions are equivalent:

1. \(A\) is a \(\phi\)-2-absorbing primary ideal of \(S\).
2. For each elements \(a\) and \(b\) of \(S\) if \(ab \in S - A\), then \((A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\) for some positive integer \(n\).

**Proof.** First assume that (1) holds. Let \(a, b\) and \(c\) be any elements of \(S\) such that \(c \in (A : ab)\). Then we have, \(abc \in A\). If \(abc \notin \phi(A)\), then \(abc \notin A - \phi(A)\). Since \(A\) is a \(\phi\)-2-absorbing primary ideal of \(S\), we have \(ab \in A\) or \(bc \in \sqrt{A}\) or \(ac \in \sqrt{A}\).

By assumption, \(c \in \sqrt{(\sqrt{A} : a^n)}\) or \(c \in \sqrt{(\sqrt{A} : b^n)}\) for some positive integer \(n\) that is, \(c \in \sqrt{(\sqrt{A} : a^n)}\) or \(c \in \sqrt{(\sqrt{A} : b^n)}\) \(\subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\).

If \(abc \in \phi(A)\), then \(c \in (\phi(A) : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\).

Consequently, \((A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\).

Conversely, assume that (2) holds. Let \(a, b\) and \(c\) be any elements of \(S\) such that \(abc \in A - \phi(A)\). Then we have, \(c \in (A : ab)\) and \(c \notin (\phi(A) : ab)\). If \(ab \in A\), then there is nothing to prove. If \(ab \notin A\), then \((A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\) for some positive integer \(n\). Since \(c \in (A : ab)\) and \(c \notin (\phi(A) : ab)\), we have \(c \in \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}\). Therefore, \(bc \in \sqrt{A}\) or \(ac \in \sqrt{A}\) and hence \(A\) is a \(\phi\)-2-absorbing primary ideal of \(S\).

The next theorem gives the relationships between 2-absorbing primary ideals and \(\phi\)-2-absorbing primary ideals of a semigroup \(S\).

**Theorem 1.4.** Let \(\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}\) be a function and let \(\phi(A)\) be a 2-absorbing primary ideal of a semigroup \(S\). Then \(A\) is a \(\phi\)-2-absorbing primary ideal of \(S\) if and only if \(A\) is a 2-absorbing primary ideal of \(S\).

**Proof.** First assume that \(A\) is a 2-absorbing primary ideal of \(S\). Obviously, \(A\) is a \(\phi\)-2-absorbing primary ideal of \(S\).
Conversely, assume that $A$ is a $\phi$-2-absorbing primary ideal of $S$. Let $a, b$ and $c$ be any elements of $S$ such that $abc \in A$. If $abc \notin \phi(A)$, then $abc \in A - \phi(A)$. By assumption, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. Now if $abc \in \phi(A)$, then $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. In any case, we have $A$ is a $\phi$-2-absorbing primary ideal of $S$. □

In the following we shall introduce the notion of $\phi$-2-absorbing primary triple zero of a $\phi$-2-absorbing primary ideal $A$ in a semigroup $S$.

Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let $A$ be a $\phi$-2-absorbing primary ideal of a semigroup $S$ a triple $(a, b, c), a, b, c \in S$ is a $\phi$-2-absorbing primary triple zero if

1. $abc \in \phi(A)$
2. $ab \notin A$ and $bc \notin \sqrt{A}$ and $ac \notin \sqrt{A}$.

**Remark 1.2.** Note that a proper ideal $A$ of a semigroup $S$ is a $\phi$-2-absorbing primary ideal of $S$ that is not a 2-absorbing primary ideal of $S$ if and only if $A$ has a $\phi$-2-absorbing primary triple-zero $(a, b, c)$ for some $a, b, c \in S$.

Now we investigate the $\phi$-2-absorbing primary triple zero of a $\phi$-2-absorbing primary ideal $A$ in a semigroup $S$.

**Theorem 1.5.** Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let $A$ be a $\phi$-2-absorbing primary ideal of a semigroup $S$. For each elements $a, b$ and $c$ of $S$ if $(a, b, c)$ is a $\phi$-2-absorbing primary triple zero of $A$, then the following statements hold:

1. $abA \subseteq \phi(A)$
2. $aAc \subseteq \phi(A)$
3. $A^2c \subseteq \phi(A)$
4. $aA^2 \subseteq \phi(A)$.

**Proof.**

1. Suppose that $abA \not\subseteq \phi(A)$. Then there exists an element $d$ of $A$ such that $abd \notin \phi(A)$. Thus we have, $\{abc\} \cup \{abd\} \not\subseteq \phi(A)$, which implies that $\{ab\} (\{c\} \cup \{d\}) \not\subseteq A - \phi(A)$. Since $A$ is a $\phi$-2-absorbing primary ideal of $S$, we have $ab \in A$ or $b (\{c\} \cup \{d\}) \subseteq \sqrt{A}$ or $a (\{c\} \cup \{d\}) \subseteq \sqrt{A}$. Therefore, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which is a contradiction. Consequently, $abA \subseteq \phi(A)$. 

2. Suppose that \( aAc \not\subseteq \phi(A) \). Then there exists an element \( r \) of \( A \) such that \( arc \not\subseteq \phi(A) \). Since \( r \in A \), we have \( a\{(b) \cup \{r\}\}c \subseteq A \), which implies that \( a\{(b) \cup \{r\}\}c \subseteq A - \phi(A) \). In fact, since \( A \) is a \( \phi \)-2-absorbing primary ideal of \( S \), we have \( a\{(b) \cup \{r\}\}c \subseteq \sqrt{A} \) or \( ac \in \sqrt{A} \). Thus, \( ab \in A \) or \( bc \in \sqrt{A} \) or \( ac \in \sqrt{A} \), which is a contradiction. Consequently, \( aAc \subseteq \phi(A) \).

3. The proof is similar to part 2.

4. Assume that \( A^2c \not\subseteq \phi(A) \). Then there exist elements \( r, s \) of \( A \) such that \( rsc \not\subseteq \phi(A) \). Then by parts 2 and 3, \( \{abc\} \cup \{rbc\} \cup \{asc\} \cup \{rsc\} \not\subseteq \phi(A) \), which implies that \( \{(a) \cup \{r\}\} \cup \{s\} \subseteq A - \phi(A) \). In fact, since \( A \) is a \( \phi \)-2-absorbing primary ideal of \( S \), we have \( \{(a) \cup \{r\}\} \cup \{s\} \subseteq A \) or \( \{(b) \cup \{s\}\} \subseteq \sqrt{A} \) or \( \{(a) \cup \{r\}\} \subseteq \sqrt{A} \). Therefore, \( ab \in A \) or \( bc \in \sqrt{A} \) or \( ac \in \sqrt{A} \), which is a contradiction. Consequently, \( A^2c \subseteq \phi(A) \).

5. Suppose that \( aA^2 \not\subseteq \phi(A) \). Then there exist elements \( r, s \) of \( A \) such that \( ars \not\subseteq \phi(A) \). Therefore by parts 1 and 2 we conclude that \( \{abc\} \cup \{abs\} \cup \{arc\} \cup \{ars\} \not\subseteq \phi(A) \), which implies that \( a\{(b) \cup \{r\}\} \cup \{s\} \subseteq A - \phi(A) \). In fact, since \( A \) is a \( \phi \)-2-absorbing primary ideal of \( S \), we have \( a\{(b) \cup \{r\}\} \cup \{s\} \subseteq \sqrt{A} \) or \( \{(b) \cup \{s\}\} \subseteq \sqrt{A} \) or \( \{(a) \cup \{r\}\} \subseteq \sqrt{A} \). Thus, \( ab \in A \) or \( bc \in \sqrt{A} \) or \( ac \in \sqrt{A} \), which is a contradiction. Consequently, \( aA^2 \subseteq \phi(A) \).

As a simple consequence of Theorem 1.5 we give the following result.

**Corollary 1.1.** Let \( \phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\} \) be a function and let \( A \) be a \( \phi \)-2-absorbing primary ideal of a commutative semigroup \( S \). For every \( a, b, c \in S \) if \((a, b, c)\) is a \( \phi \)-2-absorbing primary triple zero of \( A \), then the following statements hold:

1. \( abA \subseteq \phi(A) \) and \( acA \subseteq \phi(A) \) and \( bcA \subseteq \phi(A) \);
2. \( aA^2 \subseteq \phi(A) \) and \( bA^2 \subseteq \phi(A) \) and \( cA^2 \subseteq \phi(A) \).

Now we arrive at one of our main theorem.

**Theorem 1.6.** Let \( \phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\} \) be a function and let \( A \) be a \( \phi \)-2-absorbing primary ideal of a commutative semigroup \( S \). Suppose that \( B \) is an ideal of \( S \) and \( a, b \in S \) such that \( abB \subseteq A \). If \((a, b, c)\) is not a \( \phi \)-2-absorbing primary triple zero of \( A \), \( \sqrt{A} \) for every \( c \in B \), then \( ab \in \sqrt{A} \) or \( bB \subseteq \sqrt{A} \) or \( ab \subseteq \sqrt{A} \).

**Proof.** Suppose that \( ab \not\subseteq \sqrt{A} \) and \( bB \not\subseteq \sqrt{A} \) and \( ab \not\subseteq \sqrt{A} \). Then there are exist elements \( d_1, d_2 \in B \) such that \( bd_1 \not\subseteq \sqrt{A} \) and \( ad_2 \not\subseteq \sqrt{A} \). If \( abd_1 \not\subseteq \phi(A) \), then \( abd_1 \in A - \phi(A) \). By assumption, \( ad_1 \in \sqrt{A} \) or \( bd_1 \in \sqrt{A} \). Next, let
$abd_1 \in \phi(A)$. By hypothesis, $ad_1 \in \sqrt{A}$ or $bd_1 \in \sqrt{A}$. Now if $abd_2 \notin \phi(A)$, then $abd_2 \in A - \phi(A)$. By the given hypothesis, $ad_2 \in \sqrt{A}$ or $bd_2 \in \sqrt{A}$. So let $abd_2 \in \phi(A)$. By given hypothesis, $ad_2 \in \sqrt{A}$ or $bd_2 \in \sqrt{A}$. In any case, we have $bd_1, bd_2 \in \sqrt{A}$. Since $abB \subseteq A$, we have $ab(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$. If $ab(\{d_1\} \cup \{d_2\}) \not\subseteq \phi(\sqrt{A})$, then $ab(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A} - \phi(\sqrt{A})$. Now by our hypothesis, $a(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$ or $b(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$, which implies that $bd_1, ad_2 \in \sqrt{A}$, which is a contradiction. Assume that $ab(\{d_1\} \cup \{d_2\}) \subseteq \phi(\sqrt{A})$. From our hypothesis, $a(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$ or $b(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$. Clearly, $bd_1 \in \sqrt{A}$ or $ad_2 \in \sqrt{A}$, which again is a contradiction. Hence $ab \in \sqrt{A}$ or $bB \subseteq \sqrt{A}$ or $aB \subseteq \sqrt{A}$.

\section*{REFERENCES}


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