UNIQUE METRO DOMINATION OF POWER OF PATHS

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\textbf{Abstract.} A dominating set \(D\) of \(G\) which is also a resolving set of \(G\) is called a metro dominating set. A metro dominating set \(D\) of a graph \(G(V,E)\) is a unique metro dominating set (in short an UMD-set) if \(|N(v) \cap D| = 1\) for each vertex \(v \in V - D\) and the minimum of cardinalities of an UMD-sets of \(G\) is the unique metro domination number of \(G\) denoted by \(\gamma_{\mu\beta}(G)\). In this paper we determine unique metro domination number of power of paths.

1. Introduction

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices \(u\) and \(v\) in a graph \(G\) is called the distance between \(u\) and \(v\) and is denoted by \(d(u, v)\). For a vertex \(v\) of a graph, \(N(v)\) denote the set of all vertices adjacent to \(v\) and is called open neighborhood of \(v\). Similarly, the closed neighborhood of \(v\) is defined as \(N[v] = N(v) \cup \{v\}\).

Let \(G(V,E)\) be a graph. For each ordered subset \(S = \{v_1,v_2,\ldots,v_k\}\) of \(V\), each vertex \(v \in V\) can be associated with a vector of distances denoted by \(\Gamma(v/S) = (d(v_1,v),d(v_2,v),\ldots,d(v_k,v))\). The set \(S\) is said to be a resolving set of \(G\) if \(\Gamma(v/S) \neq \Gamma(u/S)\) for every \(u,v \in V - S\), see [1]. A resolving set of minimum cardinality is a metric basis and cardinality of a metric basis is the metric dimension of \(G\), see [2]. The k-tuple, \(\Gamma(v/S)\) associated to the vertex

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$v \in V$ with respect to a metric basis $S$, is referred as a \textit{code generated by $S$} for that vertex $v$. If $\Gamma(v/S) = (c_1, c_2, \ldots, c_k)$, then $c_1, c_2, c_3, \ldots, c_k$ are called components of the code of $v$ generated by $S$ and in particular $c_1, 1 \leq i \leq k$, is called $i^{th}$-component of the code of $v$ generated by $S$.

A dominating set $D$ of a graph $G(V, E)$ is the subset of $V$ having the property that for each vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv$ is in $E$, see [3]. A dominating set $D$ of $G$ which is also a resolving set of $G$ is called a \textit{metro dominating set}.

A metro dominating set $D$ of a graph $G(V, E)$ is a \textit{unique metro dominating set} (in short an \textit{UMD set}) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$. Generally, if $|N(v) \cap D| = k$ for each vertex $v \in V - D$, $k \geq 1$, such a metro dominating set $D$ is called a \textit{Smarandachely distance $k$ dominating set} (Smarandachely $k$ DD-sets of $G$) and the minimum of cardinalities of the Smarandachely DD-sets of $G$ is the number of Smarandachely $k$ UDD-sets of $G$, denoted by $\gamma_{S_{\mu \beta}}^k(G)$. Particularly, if $k = 1$, i.e., the \textit{unique metro domination number of $G$} denoted by $\gamma_{\mu \beta}(G)$, see [4–6].

2. Main Results

Take $P_n, n > k$. If $k < i \leq n - k$, join $v_i$ to $v_{i-2}, v_{i-3}, \ldots, v_{i-k}$ and $v_{i+2}, \ldots, v_{i+k}$. If $i > n - k$, then join $v_i$ to $v_{i-2}, v_{i-3}, \ldots, v_{i-k}$ and all $v_j, j > i + 1$. Similarly if $i \leq k$, then join $v_i$ to $v_j, j < i - 1$ and to $v_{i+2}, \ldots, v_{i+k}$. The resulting graph is called $P^k_n$.

If $k < i \leq n - k$, then $v_i$ dominates $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-k}, v_{i+1}, v_{i+2}, \ldots, v_{i+k}$. If $|i - j| \leq 2k + 1$, then vertex $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ are dominated by $v_i$ and $v_j$.

The set $D = \{v_1, v_6, v_9\}$ is a dominating set in Figure 1 for $P^2_9$. It is also a metro dominating set. Note that $v_7$ is dominated by $v_6$ and $v_9$. Hence $D$ is not a UMD set.

The set $D = \{v_3, v_8\}$ is a dominating set for $P^2_9$ in Figure 2. All vertices are dominated uniquely by $\{v_3, v_8\}$. But code generated by $D$ to $v_4$ and $v_5$ is same. Hence $\{v_3, v_8\}$ does not resolve the vertex set $V$ of $P^2_9$ and hence $D$ is not a UMD set. If $v_1 \in D$, it dominates $v_2, v_3, \ldots, v_{k+1}$. If $v_i \in D$ and if $i < 2k + 2$, then $v_{k+1}$ is dominated by $v_1$ and $v_i$. If $i > 2k + 2$, then $v_{k+2}$ is not dominated. Further if $i = 2k + 2$ then the vertices $v_2, v_3, \ldots, v_{i-1}$ are uniquely dominated.

A vertex in $P^k_n$ can dominate a maximum of $2k + 1$ vertices.
Lemma 2.1. If $D$ is a minimal dominating set then $|D| \geq \left\lceil \frac{n}{2k+1} \right\rceil$.

However if $n = k + 1 + (2k + 1)p$, $(p \in \mathbb{N})$, then $D = \{v_{k+1}, v_{3k+2}, v_{5k+3}, \ldots v_{n-k}\}$ is a minimal dominating set and $|D| = \frac{n}{2k+1}$. Hence we have

Lemma 2.2. When $n = k + 1 + (2k + 1)p$, $p \in \mathbb{N}$, the domination number $\gamma(P_n^k) = \frac{n}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$.

Observe that when $n = (k + 1) + (2k + 1)p$, $p \in \mathbb{N}$, $D = \{v_{k+1}, v_{3k+2}, \ldots v_n\}$ dominates $V - D$ uniquely.

For any $v_i$ and $v_j$,

$$d(v_i, v_j) = \left\lceil \frac{|i - j|}{k} \right\rceil.$$ 

Consider $1 \leq i < j \leq n - k$, such that $j - i \equiv 0 \pmod{k}$. Then we get $d(v_i, v_{j+1}) = d(v_i, v_{j+2}) = \ldots = d(v_i, v_{j+k}) = \frac{j - i + k}{k}$.

If $k + 1 \leq t < i < j \leq n - k$, $2i = j + t$, $j - i \equiv 0 \pmod{k}$ and $i - t \equiv 0 \pmod{k}$, then $d(v_i, v_{i-1}) = d(v_i, v_{i-2}) = \ldots = d(v_i, v_{i-k}) = d(v_i, v_{i+1}) = d(v_i, v_{j+1}) = \ldots = d(v_i, v_{j+k}) = \frac{j - t + 2k}{2k}$. 

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**Figure 1.** $P_9^2$

**Figure 2.** $P_9^2$
Now consider the unique dominating set \( D = \{v_{k+1}, v_{3k+2}, \ldots\} \) for \( P_n^m \). The vertices \( v_i \) and \( v_j \in D, i = (k + 1) + (2k + 1)l \) and \( j = (k + 1) + (2k + 1)(l + 1) \), generate the same code to all the vertices in \( U = \{v_{i+1}, v_{i+2}, \ldots, v_{i+k}\} \) and the same code to all the vertices in \( W = \{v_{i+k+1}, v_{i+k+2}, \ldots, v_{j-1}\} \).

For example, \( \{v_{k+1}, v_{3k+2}\} \) generates the same code \((1, 2)\) for \( v_{k+2}, v_{k+3}, \ldots, v_{2k+1} \) and same code \((2, 1)\) for \( v_{2k+2}, v_{2k+3} \ldots v_{3k+1}. \)

Take \( v_h \in D, h = (k + 1) + (2k + 1)(l + 2). \) Then \( \frac{|h - i|}{k} = \frac{4k + 2}{k} = 4 + \frac{2}{k}. \)

Hence \( d(v_h, v_i) = \left\lceil 4 + \frac{2}{k} \right\rceil = 5 \) and \( d(v_h, v_{i+1}) = \left\lceil 4 + \frac{1}{k} \right\rceil = 5. \)

All other vertices of \( U \) will have the same distance 4 from \( v_h. \)

Now \( |h - (i + k + 1)| = |3k + 1|. \) Therefore \( d(v_h, v_{i+k+1}) = \left\lceil \frac{3k + 1}{k} \right\rceil = \left\lceil 3 + \frac{1}{k} \right\rceil = 4 \)

and all other vertices of \( W \) will have the same distance 3 from \( v_h. \) Hence code generated by \( \{v_i, v_j, v_k\} \) will be the same for vertices in \( U - \{v_{i+1}\} \) and is the same for vertices in \( W - \{v_{i+k+1}\}. \)

For example, the code generated by \( \{v_{k+1}, v_{3k+2}, v_{5k+3}\} \) to \( v_{k+2} \) is \((1,2,5)\) and to \( v_{2k+2} \) is \((2,1,4)\) where as same code \((1,2,4)\) is generated to \( v_{k+3}, \ldots, v_{2k+1}, \)

same code \((2,1,3)\) is generated to \( v_{2k+3}, \ldots, v_{3k+1}. \)

Now take \( v_h \in D, h = (k + 1) + (2k + 1)(l - 1). \) Then \( \frac{|h - i|}{k} = \frac{2k + 1}{k} = 2 + \frac{1}{k} \) and \( \frac{|h - (i + k)|}{k} = 3 + \frac{1}{k}. \) Therefore \( d(v_h, v_i) = 3 = d(v_h, v_{i+1}) = \ldots = d(v_h, v_{i+k-1}) \)

and \( d(v_h, v_{i+k}) = 4. \) Further \( \frac{|h - (j - 1)|}{k} = \frac{|4k + 1|}{k} = 4 + \frac{1}{k}, \)

and therefore \( d(v_h, v_{j-1}) = 5 \) and \( d(v_h, v_{j-2}) = 4 = d(v_h, v_{j-3}) = \ldots = d(v_h, v_{i+k+1}). \) Therefore code generated by \( \{v_i, v_j, v_h\} \) will be same for vertices in \( U - \{v_{i+k}\} \) which is different from the code of \( v_{i+k} \) and code generated will be the same for vertices in \( W - \{v_{i+k+1}\}, \) which is different from the code of \( v_{i+k+1}. \)

Every vertex of \( D, \) when added to \( \{v_i, v_j\}, \) it produces different code to exactly one vertex of \( U \) and exactly one vertex of \( W. \) Hence to resolve all the vertices between \( v_i \) and \( v_j, \) we require \( k - 1 \) vertices of \( D. \) Therefore minimum \( |D| = k + 1. \) For each \( l, \) \( 0 \leq l \leq k + 1, \) there are \( 2k \) vertices between \( v_i \) and \( v_j. \)

Also we have \( v_1, v_2, \ldots, v_k. \) Thus to resolve \( V - D, \) it is necessary to have at least \( k(2k) + (k + 1) + k = 2k^2 + 2k + 1 \) vertices in \( V. \) Thus we have

**Lemma 2.3.** If \( n = 2k^2 + 2k + 1, \) then \( D = \{v_{k+1}, v_{3k+2}, \ldots\} \) is a unique metro dominating set.
Further we observe, that if \( n \geq 2k^2+2k+1 \), and \( (2k+1)p-k \leq n \leq (2k+1)p-1 \), then \( D = \{v_{k+1}, v_{3k+2}, \ldots, v_{(2k+1)p-k}\} \) is a UMD set.

Also if \( n \geq 2k^2+2k+1 \) and \( (2k+1)p-2k \leq n < (2k+1)p - k \), then \( D = \{v_1, v_{2k+2}, v_{4k+3}, \ldots, v_{(2k+1)p-2k}\} \) is a UMD set.
In any case $|D| = p = \left\lceil \frac{n}{2k+1} \right\rceil$. Hence we have the following theorem.

**Theorem 2.1.** $\gamma_{\mu \beta}(P^k_n) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & \text{for } n \geq 2k^2 + k + 1 \\ n, & \text{for } n < 2k^2 + k + 1 \end{cases}$

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