TOPOLOGICAL GROUPS: VIRTUE OF PRE-OPEN SETS

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\textsc{Abstract.} In this paper, we introduce notions of $p$-topological group and $p$-irresolute topological group which are generalizations of the notion topological group. We discuss the properties of $p$-topological group with illustrated examples. Also, we prove that translation and inversion in $p$-topological group are $p$-homeomorphism.

1. INTRODUCTION

Topological group is a mathematical structure on a set which is defined by underlying two distinguished structures on that set namely group and a topology. A topological group in modern notion is defined as, a group binded with a topology such that the binary operations are continuous. Based on this, some generalizations of topological groups such as paratopological groups, semitopological groups and quasitopological groups are defined. In a finite group, all the above mentioned generalizations coincide [7]. The concepts of $S$-topological group and $s$-topological group were discussed in [1] and the theory of almost topological group was initiated in [5]. In this paper, we discuss some more

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generalizations which defined based on pre-open sets and present a new generalization of topological group called $p$-topological group.

2. Preliminaries

Throughout this paper, the pair $(A, \tau)$ denotes a group $A$ together with a topology $\tau$. For any $g \in A$, $g^{-1}$ denotes the inverse of $g$ in $A$. Let $S, T \subseteq A$, then $ST = \{s.t : s \in S, t \in T\}$. The notions pre-open, pre-closed and pre-continuous map follows [6]. For a set $C$ on a topological space $X$, the notions pre-interior and pre-closure are denoted by $\text{pint}(C)$ and $\text{pcl}(C)$ follows [6]. For a set $R$, the power set of $R$ is denoted by $\mathcal{P}(R)$ and for a topology $\tau$ on $A$, the collection of open sets, closed sets, pre-open sets are denoted by $O(A)$, $C(A)$ and $\tau_p$.

3. $p$-Topological Group

We introduce the concept of $p$-topological group and investigate its basic properties with illustrated examples in this section.

Definition 3.1. A pair $(A, \tau)$ is said to be $p$-topological group, for $m, n \in A$:
- for each open neighbourhood $K$ of $mn$, $\exists$ pre-open neighbourhoods $M$ of $m$ and $N$ of $n \ni MN \subseteq K$
- for each open neighbourhood $S$ of $g^{-1}$ $\exists$ pre-open neighbourhood $T$ of $g \ni T^{-1} \subseteq S$.

In other words, multiplication and inversion mappings are pre-continuous.

Example 3.2. Consider the group $A = (\mathbb{Z}_3, \oplus)$ with topology $\tau = \{\emptyset, \{1, 2\}, A\}$. For the topology $\tau$, $\tau_p = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{0, 1\}, \{0, 2\}, A\}$. Then $(A, \tau)$ is a $p$-topological group. Finite group with indiscrete topology is the only connected topological group. But here, the above mentioned $p$-topological group is connected.

Proposition 3.3. Let $A$ be a $p$-topological group and $S \in O(A)$. Then for any $g \in A$, $gS$ and $Sg$ are pre-open.

Proof. Let $h \in gS$, then $h = gs$ for some $s \in S$. Now, $s = g^{-1}h$ and by the pre-continuity of multiplication $\exists$ pre-open sets $M$ and $N$ of $g^{-1}$ and $h \ni MN \subseteq S$ which implies $h \in N \subseteq gS$. Hence $gS$ is pre-open. Similarly we can prove that $Sg$ is pre-open. \qed
Corollary 3.4. Let $T \in C(A)$ in a $p$-topological group $A$. Then for any $a \in A$, $aT$ and $Ta$ are pre-closed.

Proposition 3.5. Let $A$ be a $p$-topological group. Then $D \in \mathcal{T}_p$ if and only if $D^{-1} \in \mathcal{T}_p$.

Proof. Let $D \in \mathcal{T}_p$, then $\exists M \in O(A) \ni D \subseteq M \subseteq cl(D)$. Now, $D^{-1} \subseteq E^{-1} \subseteq (cl(D))^{-1}$. Since inversion is pre-continuous, we have $D^{-1}$ is pre-open and $(cl(D))^{-1}$ is pre-closure of $D^{-1}$. By using that, for a set $C$, $pcl(C) \subseteq cl(C)$ [4], we have $D^{-1} \subseteq M^{-1} \subseteq int(cl(M^{-1})) \subseteq (cl(M))^{-1} \subseteq cl(M^{-1})$. Hence $\exists int(cl(M^{-1})) \in O(A) \ni D^{-1} \subseteq int(cl(M^{-1})) \subseteq cl(D^{-1})$ and so $D^{-1}$ is pre-open. Proof of the converse is similar. □

Theorem 3.6. Let $S$ and $R$ be $p$-topological groups with $R$ is submaximal and $g$ be a pre-irresolute homomorphism at identity $e_S$. Then $g$ is pre-irresolute.

Proof. Let $n \in S$ and $M \in \mathcal{T}_p$ in $R$ containing $g(n) = m$. Since $R$ is submaximal, each pre-open set is open [3] and so $M$ is open. By Proposition 3.3, $m^{-1}M$ is pre-open in $T$ containing $e_T$. Since $\eta$ is pre-irresolute at identity $e_S$, $\exists N \in \mathcal{T}_p$ in $S$ containing $e_S \ni g(N) \subseteq m^{-1}M$. Given that $g$ is homomorphism and so $g(nN) = g(n)g(N) \subseteq M$. Hence $g$ is pre-irresolute. □

One may remind that, A bijective mapping $\mu : S \mapsto T$ is $p$-homeomorphism [2] if $\mu$ is pre-continuous and $\mu(D)$ is pre-open for every $D \in O(S)$.

Theorem 3.7. Let $S$ be a $p$-topological group and $k \in S$. Then for all $s \in S$,

(i) The mappings $\lambda_k(s) = ks$ and $\rho_k(s) = sk$ are $p$-homeomorphism.

(ii) Inversion mapping is $p$-homeomorphism.

Proof left to the reader.

Theorem 3.8. Let $R, M$ be a $p$-topological group and its subgroup,

(i) If $\exists D \in O(R)$ and $D \subseteq M$, then $M \in \mathcal{T}_p$.

(ii) If $M \in O(R)$, then it is pre-closed.

(iii) If $M \in O(R)$, then $M$ itself a $p$-topological group.

Proof left to the reader.
4. $p$-IRRESOLUTE TOPOLOGICAL GROUP AND PRE-CONNECTEDNESS

We discuss the independency of $p$-topological group from other generalization concepts of topological group. We also explore pre-connectedness properties of $p$-irresolute topological group through this section.

**Example 4.1.** Consider the group $S = (\mathbb{Z}_n, \oplus)$ with topology $\tau = \{\emptyset, \mathbb{Z}_n, \{0\}\}$. Then $(S, \tau)$ is $S$-topological and almost topological groups but not $p$-topological and $s$-topological group.

**Theorem 4.2.** Let $(S, \tau)$ be a pair satisfies $\exists$ at least one singleton is open in $S$ then $\tau$ is discrete if and only if $(S, \tau)$ is a $s$-topological group. By considering $(S, \tau)$ in Example 4.1, the above result need not be true in $S$-topological and almost topological groups.

**Example 4.3.** Consider the group $T = (\mathbb{Z}_n, \oplus)$ with the topology $\tau = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2, 1\}, \ldots, \mathbb{Z}_n\}$. Then $(T, \tau)$ is $S$-topological and an almost topological groups. But $(T, \tau)$ is not $p$-topological and $s$-topological groups.

**Example 4.4.** Consider $(S, \tau)$ in Example 3.2, which is a $p$-topological group. Here $(S, \tau)$ is an almost topological group but not $s$-topological group.

**Definition 4.5.** The pair $(T, \tau)$ is said to be $p$-irresolute topological group if binary operations are pre-irresolute.

A topological space $Y$ is **pre-connected** [8] if $Y \neq E \cup F$, where $E, F$ are two disjoint non-empty pre-open sets.

**Theorem 4.6.** If $A$ is a pre-connected, $p$-irresolute topological group and $H$, a discrete invariant subgroup of $A$, then $H \subseteq Z(A)$, where $Z(A)$ denotes center of $A$.

**Proof.** Suppose $H = \{e\}$, then the result is trivial. Suppose $H$ is non-trivial. Let $g \neq e \in H$. Since, $H$ is discrete, we can find $D \in O(A)$ of $g$ in $A \ni D \cap H = \{g\}$. Now, $A$ is $p$-irresolute topological group, $\exists E \in \tau_p$ of $e$ and $E.g \in \tau_p$ of $g$ in $A \ni (E.g).E^{-1} \subset D$. Let $b \in E$. Since $H$ is an normal subgroup of $A$, we have $b.H = H.b$ which implies that $b.g \in H.b$ and so $g.e.b^{-1} \in H$. It is also clear that $b.g.b^{-1} \in E.g.E^{-1} \subset D$. Therefore, $b.g.b^{-1} \in D \cap H = \{g\}$ which implies $b.g.b^{-1} = g$. Thus, $b.g = g.b$ for each $b \in E$. Since, $A$ is pre-connected, $E^n$ with $n \in \mathbb{N}$ covers $A$. Thus, every element $a \in A$ can be written in the form...
\[ a = b_1 \cdot b_2 \cdots b_n \] where \( b_1, b_2, \ldots, b_n \in E \) and \( n \in \mathbb{N} \). Since \( g \) commutes with every element of \( E \), we have \( a \cdot g = b_1 \cdot b_2 \cdots b_n \cdot g = b_1 \cdot b_2 \cdots g \cdot b_n = \cdots = b_1 \cdot b_2 \cdots b_n = g \cdot b_1 \cdot b_2 \cdots b_n = g \cdot a \). Hence, \( g \in H \subseteq Z(A) \). □

\section*{References}


