TOTAL GRAPH OF $\mathbb{Z}_n$ AND ITS COMPLEMENT WITH RESPECT TO NIL IDEAL

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ABSTRACT. Let $\mathbb{Z}_n$ be a non-reduced commutative ring and let $N(\mathbb{Z}_n)$ denote the set of the nil elements of $\mathbb{Z}_n$. In this paper, we introduce the total graph of $\mathbb{Z}_n$ with respect to $N(\mathbb{Z}_n)$, denoted by $T(\Gamma_N(\mathbb{Z}_n))$, as a simple undirected graph with all the elements of $\mathbb{Z}_n$ as vertices and any two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$. Some properties of $T(\Gamma_N(\mathbb{Z}_n))$ and its subgraphs $T_N(\mathbb{Z}_n)$ and $T_{\overline{\mathbb{Z}_n}}(\mathbb{Z}_n)$ are studied. Also, we study some properties associated to the graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, the complement of $T(\Gamma_N(R))$.

1. INTRODUCTION

The idea of the total graph of a commutative ring $R$, denoted by $T(\Gamma(R))$, was first put forward by Anderson and Badawi [3] as a simple undirected graph having vertex set $R$ and two distinct vertices $x$ and $y$ of $T(\Gamma(R))$ are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of all the zero-divisors of $R$. One can find detailed literature on total graphs in [3-5,7,8].

P. W. Chen [6], in the year 2003, introduced a special kind of graph structure of a commutative ring $R$ whose vertex set contains all the elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all the nil elements of the ring $R$. This concept was

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later modified by Ai-Hua Li and Qi-Sheng Li [2] who defined it as an undirected simple graph $\Gamma_N(R)$ with vertex set $Z_N(R)^* = \{x \in R^* \mid xy \in N(R) \text{ for some } y \in R^* = R - \{0\}\}$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy \in N(R)$ or $yx \in N(R)$.

In this paper, we take $R = \mathbb{Z}_n$. Throughout this paper, we shall use the notation $N(\mathbb{Z}_n)$ to denote the set of all the nil elements of the ring $\mathbb{Z}_n$. That is, $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 = 0\}$. For any commutative ring $R$, $N(R)$ is an ideal of $R$. We call this ideal a nil ideal of the ring $R$. We define the total graph of $\mathbb{Z}_n$ with respect to $N(\mathbb{Z}_n)$, denoted by $T(\Gamma_N(\mathbb{Z}_n))$, as a simple, undirected graph whose vertex set contains all the elements of $\mathbb{Z}_n$ and any two distinct vertices $x$ and $y$ of $T(\Gamma_N(\mathbb{Z}_n))$ are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$. Let $T_{N(\mathbb{Z}_n)}$ and $T_{N(\mathbb{Z}_n)}^{\overline{N(\mathbb{Z}_n)}}$ denote the induced subgraphs of $T(\Gamma_N(\mathbb{Z}_n))$ whose vertex sets are $N(\mathbb{Z}_n)$ and $\overline{N(\mathbb{Z}_n)}$ respectively, where $\overline{N(\mathbb{Z}_n)} = \mathbb{Z}_n - N(\mathbb{Z}_n)$. Also, the complement of the total graph $T(\Gamma_N(\mathbb{Z}_n))$, denoted by $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, is the simple undirected graph whose vertex set is $\mathbb{Z}_n$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in \mathbb{Z}_n - N(\mathbb{Z}_n)$.

2. Preliminaries

For any graph $G$, the diameter of $G$, denoted by $diam(G)$ is given by $diam(G) = sup\{d(x, y) : \text{where } x \text{ and } y \text{ are distinct vertices of } G\}$ and $d(x, y)$ is the length of the shortest path joining $x$ and $y$. The girth of a graph $G$, denoted by $gr(G)$, is the length of the shortest cycle in $G$. If $G$ contains no cycles, then $gr(G) = \infty$. A graph $G$ is said to be Eulerian if and only if the degree of each of its vertices is even. A non-empty subset $S$ of the set of all the vertices $V$ of a graph is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number $\gamma$ of a graph $G$ is defined to be the minimum cardinality of a dominating set in $G$ and the corresponding dominating set is called a $\gamma$–set of $G$.

A ring $R$ is said to be non-reduced if it contains at least one non-zero nil element. Otherwise it is said to be reduced.

3. The basic structure of $T(\Gamma_N(\mathbb{Z}_n))$

For any non-reduced $\mathbb{Z}_n$, the total graph $T(\Gamma_N(\mathbb{Z}_n))$ of $\mathbb{Z}_n$ with respect to its nil ideal $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 \equiv 0 \pmod{n}\}$ is a simple, undirected graph
having vertex set $Z_n$ and any two distinct vertices $x$ and $y$ of $T(\Gamma_N(Z_n))$ are adjacent if and only if $x + y \in N(Z_n)$.

**Proposition 3.1.** Let $Z_n$ be non-reduced and let $n$ be odd. Suppose that $\exists$ some $m \in Z_n - N(Z_n)$ such that $2m \in N(Z_n)$. Then $2m = n_1$, for some $n_1 \in N(Z_n)$,

$$m = \frac{n_1}{2} \begin{cases} \in N(Z_n) & \text{if } n_1 \text{ is even} \\ \notin Z_n & \text{if } n_1 \text{ is odd} \end{cases}.$$ 

In both the cases, we get a contradiction. Thus for any non-reduced $Z_n$ and for any odd $n$, $\exists$ no $m \in Z_n - N(Z_n)$ such that $2m \in N(Z_n)$.

Again, let $Z_n$ be non-reduced and let $n$ be even. Since $Z_n$ is non-reduced, so either $n = 2^k$ for some $k > 1$, or $n = 2^s \cdot p_1^{r_1} \cdot p_2^{r_2} \cdots p_s^{r_s}$, where at least one $r, r'_s > 1$ (since $Z_n$ is non-reduced).

(i) Let $n = 2^k$. Then $\exists$ some $m = 2^k - 2^{\frac{k-1}{2}} \in Z_{2^k} - N(Z_{2^k})$ such that $m + m = 2m \in N(Z_{2^k})$.

(ii) Let $n = 2^s \cdot p_1^{r_1} \cdot p_2^{r_2} \cdots p_s^{r_s}$. Then $\exists$ some $m = 2^{\left\lfloor \frac{r_1+1}{2} \right\rfloor + 1} \cdot p_1^{\left\lfloor \frac{r_2+1}{2} \right\rfloor} \cdots p_s^{\left\lfloor \frac{r_s+1}{2} \right\rfloor} \in Z_n - N(Z_n)$ such that $m + m = 2m \in N(Z_n)$.

Thus for any non-reduced $Z_n$ and for any even $n$, $\exists$ some $m \in Z_n - N(Z_n)$ such that $2m \in N(Z_n)$.

## 4. Main Results

For $R = Z_n$, the set $N(R)$ is an ideal of $R$. Since $N(R)$ is closed under addition, so for any distinct elements $x, y \in N(R), x + y \in N(R)$.

Throughout this section, we use the notation $|N(Z_n)| = \alpha$ and $|Z_n - N(Z_n)| = \beta$.

**Theorem 4.1.** Let $R = Z_n$ be non-reduced and $N(Z_n)$ be the set of all the nil elements of $Z_n$. Then $T_{N(Z_n)}$ is a complete subgraph of $T(\Gamma_N(Z_n))$ and $T_{N(Z_n)}$ is disjoint from $T_{N(Z_n)/\beta}$.

**Theorem 4.2.** Let $R = Z_n$ and let $|R| = \alpha$ and $|R - N(R)| = \beta$. Then

1. If $|R|$ is odd, then $T_{N(R)/\beta}$ is the disjoint union of $\frac{\beta}{2\alpha}$ complete bipartite graphs $K_{\alpha,\alpha}$.

2. If $|R|$ is even, then $T_{N(R)/\beta}$ is the disjoint union of the complete graph $K_{\alpha}$ and $\frac{\beta - \alpha}{2\alpha}$ complete bipartite graphs $K_{\alpha,\alpha}$. 
Theorem 4.3. [1] Let $R = \mathbb{Z}_n$, $|\text{N}(R)| = \alpha$ and $|R - \text{N}(R)| = \beta$. Then

1. If $|R|$ is odd, then $T(\Gamma_{\text{N}}(R))$ is the disjoint union of the complete graph $K_\alpha$ and $\frac{\beta}{2\alpha}$ complete bipartite graphs $K_{\alpha,\alpha}$.
2. If $|R|$ is even, then $T(\Gamma_{\text{N}}(R))$ is the disjoint union of two complete graphs $K_\alpha$ and $\frac{\beta - \alpha}{2\alpha}$ complete bipartite graphs $K_{\alpha,\alpha}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$T(\Gamma(\mathbb{Z}_{16}))$}
\end{figure}

Theorem 4.4. For $R = \mathbb{Z}_n$, let $x$ be any vertex of $T(\Gamma_{\text{N}}(R))$. Then

\[ \deg(x) = \begin{cases} 
\alpha & \text{if } x \in R - \text{N}(R) \text{ such that } 2x \notin \text{N}(R) \\
\alpha - 1 & \text{if } x \in \text{N}(R) \text{ or } x \in R - \text{N}(R) \text{ such that } 2x \in \text{N}(R).
\end{cases} \]

Proof. From Theorem 4.3, we have

\[ T(\Gamma_{\text{N}}(R)) = \begin{cases} 
K_\alpha \cup (\frac{\beta - \alpha}{2\alpha})K_{\alpha,\alpha} \cup K_\alpha, & \text{if } |R| \text{ is even} \\
K_\alpha \cup (\frac{\beta}{2\alpha})K_{\alpha,\alpha}, & \text{if } |R| \text{ is odd}
\end{cases}, \]

where the unions are disjoint.

Let $x \in T(\Gamma_{\text{N}}(R))$. Clearly, two cases arise:

Case 1: $|R|$ is odd. If $x \in K_\alpha$, then $\deg(x) = \alpha - 1$. If $x \in K_{\alpha,\alpha}$, then $\deg(x) = \alpha$.

Case 2: $|R|$ is even. If $x \in K_\alpha$, then $\deg(x) = \alpha - 1$. If $x \in K_{\alpha,\alpha}$, then $\deg(x) = \alpha$. Since $x \in K_\alpha$, $\forall x \in T_{\text{N}(R)}$ or $\forall x \in R - \text{N}(R)$ such that $2x \notin \text{N}(R)$ and since $x \in K_{\alpha,\alpha}$ $\forall x \in R - \text{N}(R)$ such that $2x \in \text{N}(R)$, therefore

\[ \deg(x) = \begin{cases} 
\alpha & \text{if } x \in R - \text{N}(R) \text{ such that } 2x \notin \text{N}(R) \\
\alpha - 1 & \text{if } x \in \text{N}(R) \text{ or } x \in R - \text{N}(R) \text{ such that } 2x \in \text{N}(R).
\end{cases} \]

\[ \square \]
Theorem 4.5. The number of edges of $T(\Gamma_N(Z_n))$ are
\[
\begin{cases}
\frac{\alpha(\alpha+\beta-1)}{2} & \text{if } n \text{ is odd} \\
\frac{\alpha(\alpha+\beta-2)}{2} & \text{if } n \text{ is even}
\end{cases}
\]

Proof. Let $n$ be odd. By Theorem 4.2, $T(\Gamma_N(Z_n))$ is the disjoint union of $1$ $K_\alpha$ and $\frac{\beta}{2\alpha} K_{\alpha,\alpha}$’s. Therefore, by the Sum of Degrees of Vertices Theorem, $\alpha(\alpha-1)+\alpha\beta = 2|E|$, where $|E|$ denotes the number of edges, $\Rightarrow |E| = \frac{\alpha(\alpha+\beta-1)}{2}$.

Next, let $n$ be even. Then $T(\Gamma_N(Z_n))$ is the disjoint union of $2$ $K_\alpha$’s and $\frac{\beta-\alpha}{2\alpha} K_{\alpha,\alpha}$’s. Therefore, $\alpha(\alpha-1)+\alpha(\alpha-1)+\alpha(\beta-\alpha) = 2|E| \Rightarrow |E| = \frac{\alpha(\alpha+\beta-2)}{2}$.

Theorem 4.6. The graph $T(\Gamma_N(Z_n))$ is non-Eulerian $\forall n \in \mathbb{N}$.

Proof. From Theorem 4.4 for any $x \in T(\Gamma_N(Z_n))$,
\[
deg(x) = \begin{cases}
\alpha & \text{if } x \in R - N(R) \text{ such that } 2x \not\in N(R) \\
\alpha - 1 & \text{if } x \in N(R) \text{ or } x \in R - N(R) \text{ such that } 2x \in N(R)
\end{cases}
\]
So the graph $T(\Gamma_N(Z_n))$ contains vertices of degree $\alpha$ as well as $\alpha - 1$, which clearly have different parities. So the degree of each vertex of $T(\Gamma_N(Z_n))$ is not even and therefore $T(\Gamma_N(Z_n))$ is not an Eulerian graph.

Theorem 4.7. For any $m_1, m_2 \in Z_n - N(Z_n)$, $m_1$ is adjacent to $m_2$ if and only if every element of the coset $m_1 + N(Z_n)$ is adjacent to every element of the coset $m_2 + N(Z_n)$.

Proof. that $m_1$ is adjacent to $m_2$. Then $m_1 + m_2 \in N(Z_n)$ and thus $m_2 = z_i - m_1$, $z_i \in N(Z_n)$. The elements of the coset $m_1 + N(Z_n)$ are adjacent to the elements of the coset $(z_i - m_1) + N(Z_n)$ since for some $n_1, n_2 \in N(Z_n)$, $(m_1 + n_1) + (z_i - m_1 + n_2) = z_i + (n_1 + n_2) \in N(Z_n)$. Conversely, let each element of the coset $m_1 + N(Z_n)$ be adjacent to each element of $m_2 + N(Z_n)$. Then for some $n_1, n_2 \in N(Z_n)$, $(m_1 + n_1) + (m_2 + n_2) \in N(Z_n) \Rightarrow (m_1 + m_2) + (n_1 + n_2) \in N(Z_n) \Rightarrow m_1 + m_2 \in N(Z_n)$. Therefore, $m_1$ is adjacent to $m_2$.

Theorem 4.8. Let $R$ be a non-reduced commutative ring with unity. Then the following conditions hold:

1. Let $G$ be an induced subgraph of $T_{\overline{N(R)}}$ and let $m_1, m_2 \in G$ such that $m_1 \neq m_2$ and let $m_1$ and $m_2$ be connected by a path in $G$. Then $diam(T_{\overline{N(R)}}) \leq 2$. 

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Let $R - N(R) \neq \emptyset$. If $T_{N(R)}$ is connected and contains a cycle, then $gr(T_{N(R)}) = 3$ or $4$.

**Proof.**

(1) If $m_1$ is adjacent to $m_2$, then $d(m_1, m_2) = 1$. Let $d(m_1, m_2) > 1$ and let $m_1 - a_1 - a_2 - m_2$ be a path in $G$ between $m_1$ and $m_2$. Then $m_1 + a_1, a_1 + a_2, a_2 + m_2 \in N(R)$. Now, $m_1 + m_2 = (m_1 + a_1) - (a_1 + a_2) - (a_2 + m_2) \in N(R)$, since $N(R)$ is an ideal of $R$. Hence, $m_1$ is connected to $m_2$ by a path of length $2$. Thus, $diam(T_{N(R)}) \leq 2$.

(2) The result easily follows from Theorem 4.2 since $gr(K_\alpha) = 3$ for $\alpha \geq 3$ and $gr(K_{\alpha,\alpha}) = 4$. □

**Theorem 4.9.** Let $\mathbb{Z}_n$ be non-reduced and $N(\mathbb{Z}_n)$ be the set of all the nil elements of $\mathbb{Z}_n$. Then $diam(T(\Gamma_N(\mathbb{Z}_n))) = 2$.

**Proof.** Since the diameter of any disconnected graph is equal to the maximum diameter of its connected components, so using Theorem 4.3, since $T(\Gamma_N(\mathbb{Z}_n))$ is the disjoint union of complete and complete bipartite graphs, so $diam(T(\Gamma_N(\mathbb{Z}_n))) = diam(K_{\alpha,\alpha})$. Also, $\mathbb{Z}_n$, being non-reduced, $|N(\mathbb{Z}_n)| = \alpha \geq 2$. Consequently, $diam(T(\Gamma_N(\mathbb{Z}_n))) = 2$. □

**Theorem 4.10.** Let $f : R_1 \rightarrow R_2$ be a homomorphism. For any $m_1, m_1' \in R_1$, if the coset $m_1 + N(R_1)$ is adjacent to each element of $m_1' + N(R_1)$, then $f(m_1) + N(R_2)$ is adjacent to each element of $f(m_1') + N(R_2)$.

**Proof.** Let $m_1 + N(R_1)$ be adjacent to $m_1' + N(R_1)$. Then for some $r_1, r_1' \in N(R_1)$, $(m_1 + r_1) + (m_1' + r_1') \in N(R_1) \Rightarrow (m_1 + m_1') + (r_1 + r_1') \in N(R_1) \Rightarrow m_1 + m_1' \in N(R_1)$. $f$, being a homomorphism, preserves adjacency and, thus, $f(m_1)$ is adjacent to $f(m_1')$ in $R_2$. That is, $f(m_1) + f(m_1') \in N(R_2)$. So for some $n_1, n_1' \in N(R_2)$, $(f(m_1) + n_1) + (f(m_1') + n_1') \in N(R_2) \Rightarrow f(m_1) + N(R_2)$ is adjacent to each element of $f(m_1') + N(R_2)$. □

5. Some properties associated to $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, the complement of $T(\Gamma_N(\mathbb{Z}_n))$.

Being a complement of $T(\Gamma_N(\mathbb{Z}_n))$, the graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ contains all the elements of $\mathbb{Z}_n$ as vertices and any two distinct vertices $x$ and $y$ of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ are adjacent if and only if $x + y \in \mathbb{Z}_n - N(\mathbb{Z}_n)$. 
Theorem 5.1. Let $R = \mathbb{Z}_n$. For $x \in N(R)$ and $y \in R - N(R)$ such that $2y \in N(R)$, \{x + N(R)\} \cup \{y + N(R)\} forms a complete bipartite graph in $\overline{T(\Gamma_N(R))}$.

Proof. In the graph $\overline{T(\Gamma_N(R))}$, each element of the coset $x + N(R)$ is adjacent to each element of the coset $y + N(R)$ since $(x + n_1) + (y + n_2) = (x + y) + (n_1 + n_2) \in R - N(R)$, for some $n_1, n_2 \in N(R)$, since $x + y \in R - N(R)$. Also the elements of the coset $x + N(R)$ are not adjacent to each other because for some $n_1, n_2 \in N(R)$, $(x + n_1) + (x + n_2) = 2x + (n_1 + n_2) \in N(R)$. Also since $2y + (n_1 + n_2) \in N(R)$, for some $n_1, n_2 \in N(R)$, so the elements of the coset $y + N(R)$ are not adjacent to each other. Consequently, $\{x + N(R)\} \cup \{y + N(R)\}$ forms a complete bipartite graph in $\overline{T(\Gamma_N(R))}$. \qed

Theorem 5.2. Let $R = \mathbb{Z}_n$ and $x$ be any vertex of $\overline{T(\Gamma_N(R))}$. Then

\[
\text{deg}(x) = \begin{cases} 
n - \alpha & \text{if } x \in N(R) \text{ or } x \in R - N(R) \text{ such that } 2x \in N(R) \\
n - \alpha - 1 & \text{if } x \in R - N(R) \text{ such that } 2x \in R - N(R) \end{cases}
\]

The proof follows directly from Theorem 4.4.

Theorem 5.3. $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is never Eulerian.

The proof is straightforward since the degrees $n - \alpha$ and $n - \alpha - 1$ are of opposite parities.

Theorem 5.4. Let $R = \mathbb{Z}_n$. Then the following statements hold:

(a) If $R$ is a field such that $|R| > 2$, then $\overline{T(\Gamma_N(R))}$ contains an isolated vertex.

(b) $\overline{T(\Gamma_N(R))}$ contains no vertex of degree $n - 1$.

(c) $\overline{T(\Gamma_N(R))}$ contains no isolated vertex.

(d) $\overline{T(\Gamma_N(R))}$ contains a vertex of degree $n - 1$ if $R$ is a field.

Proof.

(a) Since $R$ is a field, so $N(R) = \{0\}$. Thus for each $x \in R$, $\exists$ a unique $y \in R$ such that $x + y = 0 \in N(R)$, i.e. $x = -y$. This gives us $\binom{n-1}{2}$ pairs of complete graphs $K_2$ and an isolated vertex $0$.

(b) For any $R = \mathbb{Z}_n$, since $1, (n - 1) \in R - N(R)$, so $|N(R)| = \alpha \leq n - 2$. For any $x \in N(R)$ or $x \in R - N(R)$ such that $2x \in N(R)$, by Theorem 4.4, $\text{deg}(x) = \alpha - 1 \leq n - 3$. For any $x \in R - N(R)$ such that $2x \in R - N(R)$, $\text{deg}(x) = \alpha \leq n - 2$. So in either case, the vertices of $\overline{T(\Gamma_N(R))}$ have degree less than $n - 1$. 

The proof is straightforward since the degrees $n - \alpha$ and $n - \alpha - 1$ are of opposite parities.
(c) Let $\overline{T(\Gamma_N(R))}$ contain an isolated vertex $x$. Then in $T(\Gamma_N(R))$, $\deg(x) = n - 1$. But that contradicts (b). Hence $\overline{T(\Gamma_N(R))}$ contains no isolated vertex.

(d) Let $R = \mathbb{Z}_n$ be a field. By result (a), since $T(\Gamma_N(R))$ contains an isolated vertex, say $x$, thus, in $\overline{T(\Gamma_N(R))}$, $\deg(x) = n - 1$. Hence the result follows. \hfill \Box

**Theorem 5.5.** For any $n > 1$ and non-reduced $\mathbb{Z}_n$, $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is always connected.

**Proof.** $N(\mathbb{Z}_n)$, being an ideal of $\mathbb{Z}_n$, all the vertices of $N(\mathbb{Z}_n)$ are adjacent to each other in $T(\Gamma_N(\mathbb{Z}_n))$ and therefore in the corresponding graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, each $x_i \in N(\mathbb{Z}_n)$ is adjacent to each $y_i \in \mathbb{Z}_n - N(\mathbb{Z}_n)$. So the graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is connected. \hfill \Box

**REFERENCES**

[1] A. MISHRA, K. PATRA: Intersection Graph of $\gamma-$sets in the total graph of $\mathbb{Z}_n$ with respect to nil ideal, Communicated.


