BI-INTERIOR IDEALS IN TGSR

A. Nagamalleswara Rao, L. N. P. Varma, G. Srinivasa Rao, D. Madhusudhana Rao, and Ch. Ramprasad

ABSTRACT. In this paper, we will discuss notation of bi-interior ideals as a generalization of quasi-ideal, bi-ideal, interior ideals and bi-interior ideals of TGSR and study the properties of bi-interior ideals of TGSR.

1. INTRODUCTION AND PRELIMINARIES

During 1950-1980, the concept of bi-ideals, quasi-ideals and interior ideals were studied by many mathematicians. In this paper, we introduced the notation of prime bi-interior ideals of TG Semi rings. G. Srinivasa Rao et.al [5–9] studied ternary semi rings. A lot of literature is available related to this work [1–4].

Let \((R, +)\) and \((\Gamma, +)\) be commutative semi groups. Then we call \(R\) a TG-semi ring \((TGS)\), if there is mapping \(R \times \Gamma \times R \times \Gamma \times R \rightarrow R\) \((imagesof (p, a, q, b, r))\) will be denoted by \(paqbr\), \(\forall p, q, r, s, t \in R\) and \(a, b, c, d \in \Gamma\):

\[
\begin{align*}
(1) \quad &pa(q + r)bs = paqbs + parbs \\
(2) \quad &(p + q)arbs = parbs + qarbs
\end{align*}
\]

\(^1\)corresponding author

2020 Mathematics Subject Classification. 16Y60, 06Y99.
Key words and phrases. Quasi-ideal, bi-ideal, bi-interior ideal, regular TGSR, bi-interior simple TGSR.

Submitted: 28.01.2021; Accepted: 12.02.2021; Published: 13.03.2021.

1183
A TGS $R$ is said to be commutative TGS, if $pqbr = parbq = qarb = qapbr = rapbq, \forall pqr \in R$ and $a, b \in \Gamma$. Let $R$ be a TGS. An element $e \in R$ is said to be unity element or neutral element if for each $p, q \in R \exists a, b \in \Gamma \ni pqbe = paeb = e$. A TGS $R$ is said to have zero element if there exists an element $0 \in R$ such that $0 + p = p$ and $0apb0 = pa0b0 = 0a0bp = 0 \forall p \in R$, $a, b \in \Gamma$. If there exists $a,b \in \Gamma \ni p = pabp$, then an element $p$ is known as an idempotent element. $R$ is said to be an TGS $R$, if each element in $R$ is an idempotent. A TGS $R$ is called a division TGS if for each non-zero element of $R$ has inverse with respect to multiplication. An element $p$ in TGS $R$ is said to be regular, if $\exists x, y$ in $R$ and $a, b, c, d$ in $\Gamma$ such that $p = pxbpcydp$. If every element in TGS $R$ is regular element, then $R$ is called regular TGSR.

**Definition 1.1.** A non-empty subset $S$ is said to be ternary sub $\Gamma - \text{semi-ring}$ of $R$, if $S$ is a sub-semi-group with respect to $+$ of $R$ and $a\alpha b\beta c \in S, \forall a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

**Definition 1.2.** A non-empty subset $I$ of a ternary $\Gamma$-semi-ring of a ternary $\Gamma$-semi-ring $R$ is said to left(lateral, right) ternary $\Gamma$-ideal of $R$, if (1) $a, b \in I \rightarrow a + b \in I$; (2) $a, b \in R, i \in I, \alpha, \beta \in \Gamma \implies a\alpha b\beta i \in I(a\alpha i\beta b \in I, i\alpha a\beta b \in \Gamma)$. An ideal $I$ is said to be ternary $\Gamma$-ideal, if it is left, lateral and right $\Gamma$ ideal of $R$.

**Example 1.** Consider the set $Z = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$ and Gamma be the set of all even numbers. Then with respect to usual addition and ternary multiplication, $Z$ is ternary Gamma semi ring.

**Example 2.** Let $Q = R$ be the set of all rational numbers and $\Gamma$ the set of all natural numbers. Define a mapping $R \times \Gamma \times R \times \Gamma \times R \rightarrow R$ by usual addition and ternary multiplication defined by $(p, a, q, b, r) = pqbr, \forall p, q, r \in R, a, b \in \Gamma$ then $R$ is a ternary $\Gamma$ semi ring.

**Definition 1.3.** Let $\phi \neq S \subseteq R$, where $R$ is a TGS. The set $S$ is said to be a TG-subsemi ring of $R$, if $(S, +)$ is a ternary sub semi group (TSSG) of $(R, +)$ and $STST \subseteq S$.

**Definition 1.4.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be a quasi-ideal (QI) of $R$, if $S$ is a TG-sub semi ring (TGSSR) of $R$ and $(ST \Gamma R \cap \Gamma ST \Gamma R \cap \Gamma R \Gamma ST \Gamma R) \cap (\Gamma R \Gamma S) \subseteq S$. 
**Definition 1.5.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be a bi-ideal (BI) of $R$, if $S$ is a (TGSSR) of $R$ and $S \Gamma R \Gamma S \Gamma R \Gamma S \subseteq S$.

**Definition 1.6.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be an interior-ideal (II) of $R$, if $S$ is a (TGSSR) of $R$ and $S \Gamma R \Gamma S \Gamma R \Gamma S \subseteq S$.

**Definition 1.7.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be a rt. (medial, lt.) ideal of $R$, if $S$ is a (TGSSR) of $R$ and $S \Gamma R \Gamma S \Gamma R \Gamma S \subseteq S$.

**Definition 1.8.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be an ideal of $R$, if $S$ is a (TGSSR) of $R$ and $S \Gamma R \Gamma S \Gamma R \Gamma S \subseteq S$.

**Definition 1.9.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ is said to be a k-ideal of $R$, if $S$ is a (TGSSR) of $R$ and $S \Gamma R \Gamma S \Gamma R \Gamma S \subseteq S$ and $p \in R$, $p + q \in S$, $q \in S$ then $p \in S$.

**Definition 1.10.** Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $P$ is said to be a bi-interior-ideal (BII) of $R$, if $P$ is a (TGSSR) of $R$ and $(R \Gamma R \Gamma P \Gamma R \Gamma R) \cap (P \Gamma R \Gamma P \Gamma R \Gamma P) \subseteq P$.

**Definition 1.11.** A TGSSR $R$ is said to be left (lateral, right) simple TGSSR, if $R$ has no proper left (lateral, right) ideal of $R$. A TGSSR $R$ is said to be simple TGSSR, if $R$ has no proper ideals. A TGSSR $R$ is said to be a bi-quasi-simple TGSSR, if $R$ has no proper bi-quasi-ideals of $R$.

**Example 3.** Consider the Tsemiring $R = \Gamma = M_{2 \times 2}(W)$ where $W = 0, 1, 2, 3, ...$. Then $R$ is a TG-semi ring with $P \alpha Q \beta S$ is the ordinary ternary multiplication of matrices, $\forall P, \alpha, Q, \beta, S \in R$.

$$U = \left\{ \begin{pmatrix} x & p \\ 0 & q \end{pmatrix} : p, q \in W \right\}$$

is a bi-ideal of $R$. Also

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & p \end{pmatrix} : p \in W \right\}$$

is a bi-ideal of $R$. 


2. BI ideals, interior ideals, bi interior ideals of TGSR

Throughout this paper $R$ is a commutative TGSR with unity element.

**Definition 2.1.** A non-empty subset $P$ of a TGSR $R$ is said to be bi-interior ideal of $R$, if $P$ is a ternary $Γ$ sub semi ring of $R$ and $(RΓRΓRΓR) ∩ (PΓRΓRΓR) ⊆ P$.

**Definition 2.2.** A TGSR $R$ is called bi interior simple TGSR if $R$ has no bi interior ideal other than $R$ itself.

**Theorem 2.1.** Let $R$ be a TGSR. Then the following are hold:

1. Every left (right, lateral) ideal is a BII of $R$.
2. Every QI is a BII of $R$.
3. If $A$, $B$ and $C$ are bi-interior ideals of $R$, then $ΑΓΒΓΓΓΓΓΑ$, $ΒΓΓΓΓΓΑΓΑΓΓΓΓΓΒΓΓΓΓΓΓΓΒ$ are BIIs of $R$.
4. Every ideal is a BII of $R$.
5. If $P$ is a BII of $R$ then $PΓRΓRΓRΓP$ are BIIs of $R$.

**Theorem 2.2.** Every BI of a TGSR $R$ is a BII of $R$.

**Theorem 2.3.** Every interior ideal of a TGSR $R$ is a BII of $R$.

**Theorem 2.4.** Let $R$ be a simple TGSR. Every BII of $R$ is a BI of $R$.

*Proof.** Given $R$ is a simple TGSR. Suppose $P$ be a BII of $R$ then $(RΓRΓRΓRΓR) ∩ (PΓRΓRΓRΓP) ⊆ P$. Since $(RΓRΓRΓRΓR)$ is and $R$ is a simple TGSR, we have $(RΓRΓRΓRΓR) = R$. Since $(RΓRΓRΓRΓR) ∩ (PΓRΓRΓRΓP) ⊆ P ⇒ ((PΓRΓRΓRΓP) ∩ P ⊆ P ⇒ ((PΓRΓRΓRΓP) ⊆ P$. Hence $P$ is a BI of $R$. □

**Theorem 2.5.** Let $R$ be a TGSR. Then $R$ is a bi-interior simple TGSR $⇔ (RΓRΓaΓRΓRΓR) ∩ (aΓRΓaΓRΓa) = R, ∀a$ in $R$

*Proof.** Given $R$ is a TGSR. Suppose $R$ is a bi-interior simple TGSR $a$ in $R$. Since $R$ is a BII of $R$, we have $(RΓRΓaΓRΓR) ∩ (aΓRΓaΓRΓa) ⊆ R$. Let $a$ be a in $R$ $⇒$ $a ∈ RΓRΓaΓRΓR$ and $a ∈ aΓRΓRΓa ⇒ a ∈ (RΓRΓaΓRΓR) ∩ aΓRΓaΓRΓa$. Hence $(RΓRΓaΓRΓRΓR) ∩ (aΓRΓaΓRΓa) = R$.

Conversely suppose that $(RΓRΓaΓRΓRΓR) ∩ (aΓRΓaΓRΓa) = R, ∀a$ in $R$. Let $P$ be a BII of the TGSR $R$ and $a ∈ P$. Then $R = (RΓRΓaΓRΓRΓR) ∩ (aΓRΓaΓRΓa) ⊆ (RΓRΓaΓRΓRΓR) ∩ PΓRΓRΓRΓRΓP ⊆ P$. Therefore $R = P$. Thus $R$ is a bi-interior simple TGSR. □
Similarly it is easy to prove that $E = A\Gamma C T D$ is a BII of $R$. Let $E$ be a BII of the TGSR $S$ such that $E \subseteq P$.

Theorem 2.6. If $D$ is a minimal left ideal, $A$ is a minimal right ideal and $C$ is a lateral ideal of a TGSR $R$, then $P = A\Gamma C T D$ is a minimal BII of $R$.

Proof. Clearly $P = A\Gamma C T D$ is a BII of $S$. It is enough if we show $P$ is a minimal BII of $R$. Let $E$ be a BII of the TGSR $S$ such that $E \subseteq P$.

Similarly it is easy to prove that $ETR G E \subseteq RTR G P = RTR G (A\Gamma C T D) \subseteq D$, since $D$ is a right ideal of $R$. If $C$ is a right ideal, then $P$ is a BII of $R$.

Theorem 2.7. The intersection of a BII $P$ of a TGSR $R$ and a TGSSR $Q$ of $R$ is a BII of $R$.

Theorem 2.8. Let $A$, $C$ and $D$ be TGSSRs of a TGSR $R$ and $P = A\Gamma D \Gamma C$. If $A$ is a left ideal, then $P$ is BII of $R$.

Proof. Suppose $A$, $C$ and $D$ be TGSSRs of a TGSR $R$, $P = A\Gamma D \Gamma C$ and $A$ is a left ideal of TGSR $R$.

Consider $P\Gamma R \Gamma P\Gamma R \Gamma P = (A\Gamma D \Gamma C)\Gamma R \Gamma (A\Gamma D \Gamma C) \Gamma R \Gamma (A\Gamma D \Gamma C) \subseteq (A\Gamma D \Gamma C)\Gamma (A\Gamma D \Gamma C) \Gamma (A\Gamma D \Gamma C) \subseteq A\Gamma D \Gamma C = P \Gamma (P\Gamma R \Gamma P\Gamma R \Gamma P) \cap P \Gamma R \Gamma P\Gamma R \Gamma P \subseteq P$. Hence $P$ is a BII of $R$.

Remark 2.1. Let $A$, $C$ and $D$ be TGSSRs of a TGSR $R$ and $P = A\Gamma D \Gamma C$. If $C$ is a right ideal, then $P$ is a BII ideal of $R$.

Remark 2.2. Let $A$, $C$ and $D$ be TGSSRs of a TGSR $R$ and $P = A\Gamma D \Gamma C$. If $D$ is a lateral ideal, then $P$ is a BII of $R$.

Theorem 2.9. Let $R$ be a TGSSR and $T$ be a TGSSR of $R$. Every TGSSR of $T$ containing $(TT R \Gamma TT R \Gamma T) \cup (R \Gamma R \Gamma TT R \Gamma R)$ is a BII of $R$.

Proof. Let $P$ be a TGSSR of $T$ containing $(TT R \Gamma TT R \Gamma T) \cup (R \Gamma R \Gamma TT R \Gamma R)$. Now we show that $P$ is a BII of $R$. Consider $(P\Gamma R \Gamma P\Gamma R \Gamma P) \subseteq (TT R \Gamma TT R \Gamma T) \subseteq (TT R \Gamma TT R \Gamma T) \cup (R \Gamma R \Gamma TT R \Gamma R) \subseteq P$.

Hence $(TT R \Gamma TT R \Gamma T) \cup (R \Gamma R \Gamma TT R \Gamma R) \subseteq P$. Thus $P$ is a BII of $R$. 

□
Definition 2.3. Let \( R \) be a TGS. An element \( p \in R \) is said to be an \textit{regular element} if there exists \( x, y \in R \) such that \( p = pxbycypd \). Every element in TGS is a \textit{regular element} then \( R \) is a known as a \textit{Regular TGS}.

Theorem 2.10. Let \( R \) be a regular TGSR. Then every BII of \( R \) is an ideal of \( R \).

\textbf{Proof.} Given \( R \) is a regular element. Let us suppose \( P \) be BII of \( R \). Now we show that \( P \) is an ideal of \( R \). Since \( P \) is an II of \( R \), we have \((\Gamma R \Gamma T \Gamma T \Gamma \Gamma T) \cup (\Gamma \Gamma R \Gamma T \Gamma T \Gamma R) \subseteq P\). Consider \( \Gamma \Gamma R \Gamma R \subseteq P \Gamma \Gamma T \Gamma R P \) and \( \Gamma \Gamma R \Gamma R \cup R \Gamma R \Gamma P \Gamma R \Gamma R \Rightarrow P \Gamma R \Gamma R \subseteq TT \Gamma R \Gamma T \Gamma R \Gamma T \cup (R \Gamma R \Gamma T \Gamma R \Gamma R) \subseteq P \). Similarly, it is easy prove that \( \Gamma \Gamma R \Gamma P \subseteq P \Gamma R \Gamma R \subseteq TT \Gamma R \Gamma T \Gamma T \Gamma T \cup (R \Gamma R \Gamma T \Gamma R \Gamma R \Gamma R) \subseteq P \). Hence \( P \) is an ideal of \( R \).

\( \square \)

Theorem 2.11. Let \( R \) be a TGSR. Prove that the following statements are equivalent:

\begin{enumerate}
    \item \( R \) is a bi-interior simple TGSR.
    \item \( R \Gamma R \Gamma a = R, \forall a \in R \).
    \item \( a \rightarrow R, \forall a \in R \) and where \( a \rightarrow \) is the smallest bi-interior ideal generated by \( a \).
\end{enumerate}

\textbf{Proof.} Given \( R \) is a TGSR. To show (1) \( \Rightarrow \) (2): Suppose \( R \) is a bi-interior simple TGSR and \( a \in R \) and \( P = R \Gamma R \Gamma a \Rightarrow P \) is a left ideal of \( R \). By theorem 3.4, \( P \) is a BII of \( R \). Clearly \( P \subseteq R \) and let \( x \in R \Rightarrow x = xaxb \in R \Gamma R \Gamma a \Rightarrow R \subseteq P \) therefore \( P = R \). Hence \( \Gamma R \Gamma a = R, \forall a \in R \). To show (2) \( \Rightarrow \) (3) : Suppose \( \Gamma R \Gamma a = R, \forall a \in R \). Consider \( \Gamma R \Gamma a \subseteq a \subseteq R \) and \( R \subseteq a \subseteq R \). Therefore \( a \rightarrow R \) \( \Rightarrow \) (1) : Suppose \( \Gamma R \Gamma a \subseteq a \subseteq R \), \( \forall a \in R \). Let \( P \) be a BII and \( a \in P \) then \( a \subseteq P \subseteq R \Rightarrow R \subseteq P \subseteq R \). Therefore, \( P = R \). Hence \( R \) is a bi-interior simple TGSR.

\( \square \)

Theorem 2.12. If \( P \) is a BII of a TGSSR \( T \), \( T \) is a TGSSR of \( R \) and \( T \subseteq P \) such that \( \Gamma T \Gamma T \Gamma T \) is a ternary sub-semi-group of the ternary semi-group \((R, \cdot)\), then \( \Gamma T \Gamma T \Gamma T \) is a BII of \( R \).

\textbf{Proof.} Let \( P \) be a BII of TGSR \( R \), \( T \) be a TGSSR of \( R \) and \( T \subseteq P \) such that \( \Gamma T \Gamma T \Gamma T \) is a ternary sub-semigroup of the ternary semi-group \((R, \cdot)\). Now we show that \( \Gamma T \Gamma T \Gamma T \) is a BII of \( R \). Clearly \( \Gamma T \Gamma T \Gamma T \Gamma P \subseteq R \Gamma R \Gamma R \subseteq \Gamma R T \Gamma T \Gamma T \Gamma T \cup (R \Gamma R \Gamma T \Gamma R \Gamma R) \subseteq P \Rightarrow \Gamma T \Gamma T \Gamma T \Gamma T \subseteq \Gamma T \Gamma T \).
Hence $PTHTT$ is a TGSSR of $R$.

Also, $RΓRΓPTHTTΓRΓR ⊆ RΓRΓPTHTTΓRΓR$ and $(PTHTT)ΓRΓRΓ(PTHTT) ⊆ PTHTTΓRΓRGamma ⇒ (RΓRΓPTHTTΓRΓR)∩ (RΓRΓPTHTTΓRΓR) ⊆ (RΓRΓP) ∩ (RΓRΓR) \subseteq P$.

Hence $PTHTT$ is a BII of the TGSR of $R$. □

**Theorem 2.13.** Let $P$ be a BI of a TGSR $R$ and $Q$ be an interior ideal of $R$. Then $P \cap Q$ is a BII of $R$.

**Proof.** Suppose $P$ be a BI of a TGSSR $R$ and $Q$ be an interior ideal of $R$. Now we show that $P \cap Q$ is a BII of $R$. By the known theorem, $P \cap Q$ is a TGSSR of $R$. Also, $(P \cap Q)ΓRΓ(P \cap Q)ΓRΓR(P \cap Q) \subseteq PTHTTΓRΓR \subseteq P$ and $RΓRΓ(P \cap Q)ΓRΓR \subseteq RΓRΓQΓRΓR \subseteq Q$.

Here $(P \cap Q)ΓRΓ(P \cap Q)ΓRΓR(P \cap Q)ΓRΓR \subseteq (P \cap Q)$.

Hence $P \cap Q$ is a BII of $R$. □

**Theorem 2.14.** Let $R$ be a TGSR. If $R = RΓRΓa, \forall a \in R$. Then every BII of $R$ is a QI of $R$.

**Theorem 2.15.** If $P$ is a minimal BII of a TGSSR $R$, then any two non-zero elements of $P$ generate the same right (left, lateral) ideal of $R$.

**Theorem 2.16.** Let $R$ be a regular TGSR. Then $P$ is a BII of $R$ ⇔ $(RΓRΓPTHTTΓRΓR) \cap (RΓRΓPTHTTΓRΓR) = P, \forall BIIs Pof R$.

**Proof.** Given $R$ is a regular TGSR. Suppose $P$ be a BII of TGSR $R$ and $a \in P$. Now we show that $(RΓRΓPTHTTΓRΓR) \cap (RΓRΓPTHTTΓRΓR) = P$. Since $P$ is a BII of TGSR $R$, we have $(RΓRΓPTHTTΓRΓR) \cap (RΓRΓPTHTTΓRΓR) \subseteq P$. Also, $RΓRΓa \subseteq RΓRΓP$. Let $a \in P$ then $a$ is a regular element, because $R$ is a regular TGSR $⇒ \exists x, y \in R, α, β, γ \in Γ \ni a = ααxβyγa \in RΓRΓPTHTTΓRΓR \Rightarrow a \in RΓRΓPTHTTΓRΓR \cap (RΓRΓPTHTTΓRΓR) ⇒ P \subseteq RΓRΓPTHTTΓRΓR \cap (RΓRΓPTHTTΓRΓR)$. Hence $PTHTTΓRΓRΓR \cap (RΓRΓPTHTTΓRΓR) = P$.

Conversely, assume that $PTHTTΓRΓRΓR \cap (RΓRΓPTHTTΓRΓR) = P$, for all $BIIs Pof R$. Clearly $PTHTTΓRΓRΓR \cap (RΓRΓPTHTTΓRΓR) \subseteq P$, we have $P$ is a BII of $R$. □

**Theorem 2.17.** Let $R$ be a TGSR. If $P$ is a BII of $R$ and $P$ is a regular TGSSR of $R$, then any BII of $P$ is a BII of $R$. 
Proof. Given $R$ is a TGSR. Let $P$ be a BII of $R$ and it is a regular TGSSR of $R$. Suppose $Q$ be a BII of $P$. Now we show that $Q$ is a BII of $R$. By the theorem 3.21, $(Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) = Q$. Since $Q \subseteq P$ and $P \subseteq R$, we have $(P \Gamma R \Gamma P \Gamma R) \cap (P \Gamma R \Gamma P \Gamma R) \subseteq (Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) = Q \subseteq P \Rightarrow (P \Gamma R \Gamma P \Gamma R) \cap (P \Gamma R \Gamma P \Gamma R) \subseteq P$. \hfill $\Box$

Theorem 2.18. Let $R$ be a TGSR. Then $R$ is a bi-interior simple TGSR, if, and only if,

$$(R \Gamma R \Gamma a \Gamma R \Gamma R) \cap (a \Gamma R \Gamma a \Gamma R \Gamma a) = R, \forall a \in R.$$

Theorem 2.19. Let $R$ be a TGSR and $P$ be a BII of $R$. Then $P$ is a minimal BII of $R$ if, and only if $P$ is a bi-interior simple TGSR of $R$.

Proof. Given $R$ is a TGSR and $P$ is a BII of $R$. Let $Q$ be a BII of $P$. Now we show that $P$ is a bi-interior simple TGSR of $R$. Since $Q$ is a BIs of $P$, we have $(Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) \subseteq Q \Rightarrow (Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P)$ is a BII of $R$. Since $P$ is a minimal BII of $R$, we have $(Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) = P \Rightarrow P = (Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) \subseteq Q \Rightarrow P = Q$. Hence $P$ is a minimal BII of $R$.

Conversely suppose that $P$ is a bi-interior simple TGSR of $R$. Now we show that $P$ is a minimal BII of $R$. Let $Q$ be a BII of $R$ and $Q \subseteq P$. It is enough to show $P = Q$. Here $(Q \Gamma P \Gamma Q \Gamma P) \cap (P \Gamma P \Gamma Q \Gamma P) \subseteq (Q \Gamma R \Gamma Q \Gamma R) \cap R \Gamma R \Gamma P \Gamma R) \subseteq Q$, because $Q$ is a BII of $R$. $(Q \Gamma R \Gamma Q \Gamma R) \cap (R \Gamma R \Lambda Q \Gamma R) \subseteq (P \Gamma R \Gamma Q \Gamma R) \cap (R \Gamma R \Gamma P \Gamma R) \subseteq P \Rightarrow (Q \Gamma R \Gamma Q \Gamma R) \cap (P \Gamma P \Gamma Q \Gamma P) \subseteq (P \cap Q) \subseteq Q \Rightarrow P = Q$. Hence $P$ is a minimal BII of $R$. \hfill $\Box$

Theorem 2.20. The intersection of BIs $\{P_i : i \in \Delta\}$ of a TGSR is a BII of $R$.

Proof. Let $P = \cap_{i \in \Delta} P_i$, where $P_i$ is a BII of $R$. Now we show that $P$ is a BII of $R$. By the known theorem, $P$ is a TGSSR $R$. Since $P$ is a BII of $R$, we have $(R \Gamma R \Gamma P \Gamma R \Gamma R) \cap (R \Gamma R \Gamma P \Gamma R \Gamma P) \subseteq P_i$, for each $P_i$ and $i \in \Delta \Rightarrow (R \Gamma R \Gamma \cap P \Gamma R \Gamma R) \cap (R \Gamma R \Gamma P \Gamma R \Gamma P) \subseteq \cap_{i \in \Delta} P_i \Rightarrow ((R \Gamma R \Gamma P \Gamma R \Gamma R) \cap R \subseteq P \Rightarrow ((R \Gamma R \Gamma P \Gamma R \Gamma P) \subseteq P$. Hence $P$ is a BII of $R$. \hfill $\Box$
Definition 2.4. An element 

\[ a = \alpha \alpha \alpha \alpha a \]

is said to be \( \alpha \)-idempotent element, if \( a = \alpha \alpha \alpha \alpha a \). A ternary TGSR \( R \) is said to be \( \alpha \) – idempotent TGSR, if every element of \( R \) is \( \alpha \) – idempotent. An element \( a \) of a TGSR \( R \) is said to be \( (\alpha, \beta) \) – idempotent element, if \( a = \alpha \alpha \alpha \beta a \).

Theorem 2.21. Let \( P \) be a BII of a TGSR \( R \), \( e \) be \( (\alpha, \beta) \) – idempotent element and \( e \Gamma e P \subseteq P \). Then \( e \Gamma e P \) is a BII of \( R \).

Proof. Given \( R \) is a BII of a TGSR \( R \). Suppose \( a \in P \cap (e \Gamma R \Gamma R) \Rightarrow a \in P \) and \( a = e \alpha \beta \gamma z \), where \( \alpha, \beta, \gamma \in \Gamma \) and \( y, z \in R \). Consider \( a = e \alpha \beta \gamma z = (e \gamma e \delta) \alpha \beta \gamma z = (e \gamma e \delta) e \alpha \beta \gamma z = e \gamma e \delta a \in e \Gamma e P \). Therefore \( P \cap (e \Gamma R \Gamma R \subseteq e \Gamma e P \subseteq P \) and \( e \Gamma e P \subseteq e \Gamma R \Gamma R \) then \( e \Gamma e P = e \Gamma R \Gamma R \). Hence \( e \Gamma R \Gamma R \) is a BII of \( R \).

Theorem 2.22. Let \( R \) be a TGSR and \( e \) be a \( \alpha \)-idempotent. Then \( e \Gamma R \Gamma R, R \Gamma e \Gamma R \) and \( R \Gamma R \Gamma e \) are BII of \( R \).

Theorem 2.23. Let \( e \) and \( f \) be a \( \alpha \)-idempotent and a \( \beta \)-idempotent of TGSSR \( R \) respectively. Then, \( e \Gamma R \Gamma R \Gamma f \) is a BII of \( R \).

Theorem 2.24. Let \( P \) be a TGSSR of a regular TGSR \( R \). Then \( P \) can be expressed as \( P = K \Gamma M \Gamma L \), where \( K \) is a right ideal, \( M \) is a lateral ideal and \( L \) is a left ideal of \( R \).

Proof. Given \( P \) is a TGSSR of a regular TGSR \( R \). Suppose \( P = K \Gamma M \Gamma L \), where \( K \) is a right ideal, \( M \) is a lateral ideal and \( L \) is a left ideal of \( R \). Now we show that \( P \) is a BII of \( R \). Consider \( P \Gamma R \Gamma R \Gamma P = (K \Gamma M \Gamma L) \Gamma R \Gamma R \Gamma (K \Gamma M \Gamma L) \subseteq (K \Gamma M \Gamma L) = P \).

Consider \( (R \Gamma R \Gamma R \Gamma R \Gamma R) \cap (R \Gamma R \Gamma R \Gamma R \Gamma P) \subseteq P \Gamma R \Gamma R \Gamma R \Gamma P \subseteq K \Gamma M \Gamma L = P \). Hence \( P \) is a BII of \( R \).

Conversely, suppose that \( P \) is a BII of \( R \). Now we show that \( P \) can be expressed as \( P = K \Gamma M \Gamma L \), where \( K \) is a right ideal, \( M \) is a lateral ideal and \( L \) is a left ideal of \( R \). Since \( P \) is a BII of \( R \) by the known Theorem 3.22, \( (R \Gamma R \Gamma R \Gamma R \Gamma R) \cap (R \Gamma R \Gamma R \Gamma R \Gamma P) = P \). Let us take \( K = e \Gamma R \Gamma L \), \( L = R \Gamma e \Gamma R \) and \( M = R \Gamma e \Gamma L \), where \( e \) is the identity element of \( R \). Hence \( R \Gamma R \Gamma R \) is a right ideal of \( R \), \( L = R \Gamma e \Gamma P \) is a left ideal of \( R \) and \( M = R \Gamma e \Gamma R \) is a lateral ideal of \( R \). Consider \( (R \Gamma R \Gamma R) \cap (R \Gamma P \Gamma R) \subseteq (R \Gamma P \Gamma R \Gamma R \Gamma P) \cap (R \Gamma R \Gamma R \Gamma R \Gamma P) \subseteq P \Rightarrow (R \Gamma R \Gamma R) \cap (R \Gamma P \Gamma R) \subseteq P \Rightarrow K \cap M \cap L \subseteq P \). Also, \( P \subseteq P \Gamma R \Gamma R = R, P \subseteq R \Gamma P \Gamma R = P, P \subseteq R \Gamma R \Gamma P = P \Rightarrow P \subseteq P \).
Theorem 2.25. Let $R$ be a TGSR. Then $R$ is regular TGSR $\iff P \cap I \cap L \subseteq P \Gamma_1 \Gamma_1 L$, for any BII $P$ lateral ideal $I$ and ideal $L$ of $R$.

Proof. Given $R$ is a TGSR. Suppose $R$ is a regular TGSR. Now we show that $P \cap I \cap L \subseteq P \Gamma_1 \Gamma_1 L$, for any BII $P$ lateral ideal $I$ and ideal $L$ of $R$. Let $x \in P \cap I \cap L \Rightarrow a \in R$ and since $R$ is a regular TGSR, we have $a = a \alpha x \beta a \gamma y \delta a$, where $x, y \in R$ and $\alpha, \beta, \gamma, \delta \in \Gamma \Rightarrow a = a \Gamma R \Gamma R a \subseteq a \Gamma R \Gamma R a \Gamma R \Gamma R a \subseteq P \Gamma R \Gamma P R \Gamma P R$.

Also $a \in a \Gamma R \Gamma R a \subseteq a \in a \Gamma R \Gamma R a \subseteq a \Gamma R \Gamma R a \Gamma R \Gamma R a \Gamma R \Gamma R a \subseteq P \Gamma R \Gamma P R \Gamma P R \Rightarrow a \in (P \Gamma R \Gamma P R \Gamma P) \cap (R \Gamma R \Gamma R \Gamma P) = P \Rightarrow P \cap I \cap L \subseteq P$.

Conversely, assume that $P \cap I \cap L \subseteq P \Gamma_1 \Gamma_1 L$, for any BII $P$, ideal $I$ and left ideal $L$ of $R$. Now we show that $R$ is a regular TGSR. Let $K$ be a right ideal, $M$ be a lateral ideal and $L$ be a left ideal of $R$. Then by our assumption, $K \cap M \cap L \subseteq K \Gamma R \Gamma L \subseteq K \Gamma M \Gamma L$, we have $K \Gamma M \Gamma L \subseteq K, K \Gamma M \Gamma L \subseteq M$ and $K \Gamma M \Gamma L \subseteq L \Rightarrow K \Gamma M \Gamma L \subseteq K \cap M \cap L$.

Hence, $K \Gamma M \Gamma L = K \cap M \cap L$.

Therefore, $R$ is a regular TGSR. □

Theorem 2.26. If TGSR $R$ is a left (lateral, right) simple TGSR, then every BII of $R$ is a right (lateral, left) ideal of $R$.

Proof. Let $P$ be a BII of the left simple TGSR $R$. Then $R \Gamma R \Gamma P$ is a left ideal of $R$ and $R \Gamma R \Gamma P \subseteq R$ and clearly $R \subseteq R \Gamma R \Gamma P$. Then $R = R \Gamma R \Gamma P$. $R \Gamma R \Gamma R \Gamma R = R \Gamma R \Gamma R \subseteq R$ and $P \Gamma R \Gamma R \Gamma R \Gamma P \subseteq P \Gamma R \Gamma R$; $(P \Gamma R \Gamma R \Gamma R \Gamma P) \cap (R \Gamma R \Gamma R \Gamma R \Gamma P) = R \cap (P \Gamma R \Gamma R \Gamma P)$.

Also, $P \Gamma R \Gamma R \subseteq P \Gamma R \Gamma R \Gamma R \Gamma P \subseteq (P \Gamma R \Gamma R \Gamma P) \cap (R \Gamma R \Gamma R \Gamma P) \subseteq P$. Hence, every BII is a right ideal of $R$.

Similarly, we can prove for the right simple TGSR $R$.

The proof is completed. □

Theorem 2.27. Let $P$ be a TGSSR of a TGSR $R$. If $P$ is a BII of $R$, then $P$ is a left bi-quasi ideal of $R$.

Theorem 2.28. Let $P$ be a TGSSR of a TGSR $R$. If $P$ is a BII of $R$, then $P$ is a right (lateral) of $R$. 
Theorem 2.29. Let $P$ be a BII of $R$ and $Q$ be a non-empty subset of $P$ such that $P \Gamma Q \Gamma Q$ is a TGSSR of $R$, then $P \Gamma Q \Gamma Q$ is a BII of $R$.

Definition 2.5. An element $a$ of a ternary semi ring $R$ is said to be invertible in $R$, if there exists an element $b \in R$ (called the ternary semi ring inverse of $a$) such that $abt = bat = tba = tab = bta = t, \forall t \in R$. An element $a$ of a ternary gamma semi ring $R$ is said to be invertible in $R$, if there exists an element $b \in R$ (called the ternary gamma semi ring inverse of $a$) such that $a \alpha b \beta t = b \alpha a \beta t = t \alpha a \beta b = a \alpha t \beta b = b \alpha t \beta a = t, \forall t \in R, \alpha, \beta \in \Gamma$.

Definition 2.6. A ternary gamma semi ring $R$ with $|R| \geq 2$ is said to be a ternary division gamma semi ring, if every non-zero element of $R$ is invertible.

Theorem 2.30. Every ternary division gamma semi ring is a regular ternary gamma semi ring.

Definition 2.7. A commutative ternary division gamma semi ring $R$ is said to be a ternary gamma semi field i.e., a commutative ternary semi ring $R$ with $|R| \geq 2$, is a ternary gamma semi field, if for every non-zero element $a$ of $R$, there exists an element $b \in R, \alpha, \beta \in \Gamma$ such that $a \alpha b \beta x = x, \forall x \in R$.

Remark 2.3. A ternary gamma semi field (TGSF) $R$ has always an identity.

Theorem 2.31. If $R$ is a field TGSR, then $R$ is a bi-interior simple TGSR.

Proof. Let $P$ be a proper BII of the TGSF $R$, $t \in P$ and $0 \neq a \in P$. Since $R$ is a TGSF, $\exists b \in R$ and $\alpha, \beta \in \Gamma$ such that $a \alpha b \beta t = b \alpha a \beta t = t \alpha a \beta b = t \alpha b \beta a = a \alpha t \beta b = b \alpha t \beta a = t, \forall t \in R \Rightarrow \gamma, \delta \in \Gamma$ such that $a \gamma b \delta t = a \gamma b \delta (a \alpha b \beta t) \Rightarrow t \in P \Gamma R \Gamma R \Rightarrow R \subseteq p \Gamma R \Gamma R$ and clearly $P \Gamma R \Gamma R \subseteq R \Rightarrow R = P \Gamma R \Gamma R$.

Similarly, it is easy to prove that $R = R \Gamma R \Gamma P = R \Gamma P \Gamma R$. Consider $R = P \Gamma R \Gamma R = R \Gamma P \Gamma R = R \Gamma R \Gamma P \subseteq P \Rightarrow R \subseteq P$ and since $P \subseteq R$, we have $P = R$. Hence TGSF $R$ is a bi-interior simple TGSR.

Hence the theorem. □

Acknowledgment

The authors are highly grateful to the chief editor and referees for their careful reading, valuable suggestions and comments, which helps to improve the preparation of this paper and further research also.
REFERENCES

DEPARTMENT OF MATHEMATICS
ACHARYA NAGARJUNA UNIVERSITY
NAMBURU 522601, GUNTUR, ANDHRA PRADESH, INDIA
Email address: anmr9@gmail.com

DIVISION OF MATHEMATICS
DEPARTMENT OF SCIENCE AND HUMANITIES
V.F.S.T.R. (DEEMED TO BE UNIVERSITY)
VADLAMUDI, GUNTUR 522237, INDIA
Email address: plnvarma@gmail.com

DIVISION OF MATHEMATICS
DEPARTMENT OF SCIENCE AND HUMANITIES
V.F.S.T.R. (DEEMED TO BE UNIVERSITY)
VADLAMUDI, GUNTUR 522237, INDIA
Email address: gerinulakshmi77@gmail.com

DEPARTMENT OF MATHEMATICS
VSR AND NVR DEGREE COLLEGE
TENALI, 522017, INDIA
Email address: dmrmaths@gmail.com

DEPARTMENT OF MATHEMATICS
VVIT, NAMBURU, GUNTUR 522, INDIA
Email address: ramprasadchehu1984@gmail.com