ON ATOM-BOND CONNECTIVITY STATUS INDEX OF GRAPHS

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ABSTRACT. The atom-bond connectivity (ABC) status index of a graph is defined by V. R. Kulli as $ABC(G)=\sum_{uv\in E(G)}\sqrt{\sigma_u+\sigma_v-2}/\sigma_u\sigma_v$, where $\sigma_u$ is a status of a vertex $u \in V(G)$ and is defined as the sum of its distance from every other vertex in $V(G)$. In this paper we have obtained the bounds for the atom-bond connectivity status index. Also obtained atom-bond connectivity status index of some graphs.

1. INTRODUCTION

A topological index is a molecular structure descriptor having many applications in rationalizing the stability of linear and branched alkanes as well as the strain energy of cycloalkanes. It is a numeric numerical quantity calculated mathematically of molecule obtained from its structural graph. Estrada et.al. [12] has modified the Randić connectivity index [11] and proposed a new topological index named atom–bond connectivity (ABC) index. The atom–bond connectivity (ABC) index is widely studied [2,4–8,10,12] and for a connected

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graph $G$ it is defined as,

$$ABC \left( G \right) = \sum_{uv \in E(\mathcal{G})} \sqrt{\frac{d_u + d_v - 2}{d_ud_v}}.$$  

Where $d_u$ is the degree of vertex $u \in V(G)$.

Status \([\text{9}]\) of a vertex $u \in V(G)$ is denoted by $\sigma_u$ and is defined by the sum of its distance from every other vertex in $V(G)$.

Harmonic status index \([\text{3}]\) is defined by H.S. Ramane et. al. as

$$HS \left( G \right) = \sum_{uv \in E(G)} \frac{2}{\sigma_u + \sigma_v}.$$  

Here $\sigma_u$ is the status of vertex $u$ of $G$, $E(G)$ is the edge set. V. R. Kulli defined atom-bond connectivity status index \([\text{2}]\) of $G$ as,

$$ABCS \left( G \right) = \sum_{uv \in E(G)} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}}.$$  

2. Preliminary results

**Theorem 2.1.** \([\text{2}]\) For a complete graph $K_n$ with $n$ vertices,

$$ABCS \left( K_n \right) = \frac{n}{\sqrt{2}} \sqrt{(n - 2)}.$$  

**Theorem 2.2.** \([\text{2}]\) For a complete bipartite graph $K_{p,q}$ with $p + q$ vertices and $pq$ edges,

$$ABCS \left( K_{p,q} \right) = pq \times \sqrt{\frac{3(p + q) - 6}{2(p^2 + q^2) - 6(P + q) + (5pq + 4)}}.$$  

**Theorem 2.3.** \([\text{2}]\) For a cycle $C_n$ with $n$ vertices and $n$ edges,

$$ABCS \left( C_n \right) = \begin{cases} \frac{2}{\sqrt{2(n^2 - 4)}} & \text{if } n \text{ is even} \\ \frac{n}{2n\sqrt{2(n^2 - 5)}} & \text{if } n \text{ is odd} \end{cases}.$$  

**Theorem 2.4.** \([\text{2}]\) For a wheel graph $W_n$ with $n + 1$ vertices and $2n$ edges,

$$ABCS \left( W_n \right) = \frac{2n\sqrt{n - 2}}{(2n - 3)} + \sqrt{\frac{2n(3n - 2)}{(2n - 3)}}.$$
Theorem 2.5. [2] For a friendship graph $F_n$ with $2n + 1$ vertices and $3n$ edges,

$$ABCS(F_n) = \frac{n\sqrt{8n-6}}{4n-2} + \sqrt{\frac{n(3n-5)}{2n-1}}.$$ 

3. Obtained bounds for the atom-bond connectivity status index

Theorem 3.1. If $G$ is a connected graph having $n$ vertices and let $D$ be the diameter of $G$ then,

$$\sum_{uv \in E(G)} \sqrt{\frac{2D(n-1) - (D-1)[d(u) + d(v)] - 2}{D^2(n-1)^2 - D(n-1)[d(u) + d(v)](D-1) + d(u).d(v)(D-1)^2}}$$

$$\leq ABCS(G) \leq \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n-2 - d(u)).(2n-2 - d(v))}}.$$ 

Equality holds if and only if $\text{diam}(G) \leq 2$.

Proof.

Lower Bound: For a vertex $u \in V(G)$ of a graph $G$, $d(u)$ vertices are at distance 1 from $u$. Then the remaining vertices are $[n - 1 - d(u)]$ which are of at most diameter $D$ from $u$, and

$$\sigma(u) \leq d(u) + D(n-1 - d(u)) = D(n-1) - (D-1)d(u)$$

$$[\sigma(u) + \sigma(v)] \leq 2D(n-1) - (D-1)[d(u) + d(v)]$$

$$\sigma(u).\sigma(v) \leq [D(n-1) - (D-1)d(u)] \cdot [D(n-1) - (D-1)d(v)].$$

Therefore,

$$ABCS(G) = \sum_{uv \in E(G)} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u\sigma_v}}$$

$$\geq \sum_{uv \in E(G)} \sqrt{\frac{2D(n-1) - (D-1)[d(u) + d(v)] - 2}{D^2(n-1)^2 - D(n-1)[d(u) + d(v)](D-1) + d(u).d(v)(D-1)^2}}.$$ 

Upper Bound: Out of $n$ vertices for $u \in V(G)$, $d(u)$ vertices are at distance 1 from $u$ and the remaining $[n - 1 - d(u)]$ vertices are at the distance 2.

$$\sigma(u) \geq d(u) + 2(n-1 - d(u)) = 2n - 2 - d(u)$$

$$\sigma(v) \geq d(v) + 2(n-1 - d(v)) = 2n - 2 - d(v)$$

Therefore,
\[ ABCS(G) \leq \sum_{uv \in E(G)} \sqrt{\frac{(4n - 4) - [d(u) + d(v)] - 2}{(2n - 2 - d(u)) \cdot (2n - 2 - d(v))}}. \]

Hence, (3.1)
\[ \sum_{uv \in E(G)} \frac{2D (n - 1) - (D - 1) [d(u) + d(v)] - 2}{D^2 (n - 1)^2 - D (n - 1) [d(u) + d(v)] (D - 1) + d(u) \cdot d(v) (D - 1)^2} \leq ABCS(G) \leq \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n - 2 - d(u)) \cdot (2n - 2 - d(v))}}. \]

Equality holds when the diameter \( D \) is 1 or 2.

Conversely, let \( ABCS(G) = \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n - 2 - d(u)) \cdot (2n - 2 - d(v))}} \). Suppose \( D \geq 3 \) therefore there exist at least one pair vertices \( u \) and \( v \) such that \( d(u, v) \geq 3 \). Therefore, \( \sigma(u) \geq d(u) + 3 + 2(n - 2 - d(u)) = 2n - 1 - d(u) \). Hence,
\[ ABCS(G) \leq \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n - 2 - d(u)) \cdot (2n - 2 - d(v))}} \leq \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n - 2 - d(u)) \cdot (2n - 2 - d(v))}}. \]

This is a contradiction. Therefore \( diam(G) \leq 2 \).

**Corollary 3.1.** Let \( G \) be a connected graph having \( n \) vertices and \( m \) edges and let \( D \) be the diameter of \( G \). Let \( \delta \) be the minimum and \( \Delta \) be the maximum degree of the vertices of \( G \), then
\[ m \cdot \sqrt{\frac{2D (n - 1) - (D - 1) \cdot 2\delta - 2}{D^2 (n - 1)^2 - 2D \delta (n - 1) (D - 1) + \delta^2 (D - 1)^2}} \leq ABCS(G) \leq \sqrt{\frac{4n - 6 - 2\Delta}{(2n - 2 - 2\Delta)^2}}. \]

**Proof.** For any vertex \( u \in V(G) \), \( d(u) \geq \delta \) and \( d(u) \leq \Delta \). Therefore substituting \( [d(u) + d(v)] \geq 2\delta \) on LHS and \( [d(u) + d(v)] \leq 2\Delta \) on the RHS of equation 3.1 we obtain the result. \( \square \)
Corollary 3.2. For a connected regular graph $G$ of degree $r$ having $n$ vertices and $m$ edges and $diam(G) = D$, then,

$$m \cdot \sqrt{2D \left(n - 1\right) - 2r \left(D - 1\right)} + 2 \leq ABCS(G) \leq \sqrt{\frac{4n - 6 - 2r}{2n - 2 - 2r}}.$$ 

Equality holds if and only if $diam(G) \leq 2$.

4. Atom-bond connectivity status index of some graphs

Here we have obtained ABCS index of some graphs

Proposition 4.1. Let $W_{n+1}$ is a wheel graph with $n \geq 3$. Then,

$$ABCS \left(W_{n+1}\right) = n \times \left(\sqrt{\frac{3n - 5}{n(2n - 3)}} + \frac{4n - 8}{(2n - 3)^2}\right).$$

Proof. We give alternate proof of Theorem 2.4. Partitioning the edge set of $W_{n+1}$ in to two sets $E_1$ and $E_2$ where, $E_1 = \{uv/d(u) = n$ and $d(v) = 3\}$ and $E_2 = \{uv/d(u) = 3$ and $d(v) = 3\}$. Also, $diam(W_{n+1}) = 2$,

$$ABCS \left(W_{n+1}\right) = \sum_{uv \in E_1(G)} \sqrt{\frac{4(n+1) - 6 - (n+3)}{2(n+1) - 2 - n}} + \frac{4n - 8}{2(n+1) - 2 - 3}.$$

Thus, $ABCS \left(W_{n+1}\right) = n \times \left(\sqrt{\frac{3n - 5}{n(2n - 3)}} + \frac{4n - 8}{(2n - 3)^2}\right).$ \hfill \Box

Proposition 4.2. Let $F_n$, $n \geq 2$ be a Friendship graph. Then,

$$ABCS \left(F_n\right) = \left(2n \times \sqrt{\frac{3n - 2}{2n(2n - 1)}} + \sqrt{\frac{4n - 3}{2(2n - 1)^2}}\right) - \left(n \times \sqrt{\frac{4n - 3}{2(2n - 1)^2}}\right).$$

Proof. We give alternate proof of Theorem 2.5.

Partitioning the edge set of $F_n$ in to two sets $E_1$ and $E_2$ where, $E_1 = \{uv/d(u) = 2n$ and $d(v) = 2\}$ and $E_2 = \{uv/d(u) = 2$ and $d(v) = 2\}$. Also, $|E_1| = 2n$
and $|E_2| = n$. Also, $\text{diam}(F_n) = 2$ and $F_n$ has $2n + 1$ vertices. Therefore, by the equality part of Theorem 3.1

\[
ABCS (F_n) = \sum_{uv \in E_1(G)} \sqrt{\frac{4(2n + 1) - 6 - (2n + 2)}{2(2n + 1) - 2 - 2n}} + \sum_{uv \in E_2(G)} \sqrt{\frac{4(2n + 1) - 6 - (2 + 2)}{2(2n + 1) - 2 - 2}} = \sum_{uv \in E_1(G)} \sqrt{\frac{6n - 4}{(2n)(4n - 2)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{8n - 6}{(4n - 2)^2}} = \sum_{uv \in E_1(G)} \sqrt{\frac{2(3n - 2)}{4(n)(2n - 1)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2(4n - 3)}{4(2n - 1)^2}}.
\]

Therefore $ABCS (F_n) = 2n \times \sqrt{\frac{(3n-2)}{2n(2n-1)}} + n \times \sqrt{\frac{4n-3}{2(2n-1)^2}}$. □

**Proposition 4.3.** For a path on $n$ vertices,

\[
ABCS (P_n) = \sum_{i=1}^{n-1} \sqrt{\frac{(n - i)^2 + i^2 - 2}{\left[\frac{n^2 + n}{2} + i(i - n - 1)\right] \left[\frac{n^2 + n}{2} + (i + 1)(i - n)\right]}}.
\]

**Proof.** Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices, where $v_i$ is adjacent to $v_{i+1}$, $i = 1, 2, 3, \ldots, (n - 1)$. Therefore, $\sigma (v_i) = (i - 1) + (i - 2) + \cdots + 1 + 1 + 2 + \cdots + (n - i) = \left[\frac{n^2 + n}{2} + i(i - n - 1)\right]$ and $\sigma (u) + \sigma (v) - 2 = (n - i)^2 + i^2 - 2$.

Hence the result follows. □

5. **Atom-bond connectivity status index of subdivision graph of some graph**

**Definition 5.1.** If $G = (V, E)$ be a connected graph on $n$ vertices and $m$ edges then the subdivision graph of $G$ is denoted by $S(G)$ and defined as a graph resulting from introducing a vertex of degree two for every edge.

**Theorem 5.1.** Let $K_n$ is a complete graph on $n$ vertices. Then,

\[
ABCS [S (K_n)] = 2m \times \sqrt{\frac{7n^2 - 9n - 4}{n^2(6n^2 - 15n + 9)}}.
\]
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Proof. Partitioning the vertex set of $S(K_n)$ into two vertex set.

Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ with $|U| = n$ be the vertex set of $K_n$ and let $V = \{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of subdivision vertices with $|V| = m$. For any edge $E$ in $S(K_n)$, $E = \{uv/u \in U$ and $v \in V\}$. Therefore, every vertex $u_i \in U$ is at a distance 2 from every vertex $u_j \in U$ in $S(K_n)$. As such there are $(n - 1)$ vertices at a distance 2 from $u_i$.

Also $(n - 1)$ subdivision vertices are at distance 1 from $u_i$ and the remaining $[m - (n - 1)]$ vertices are at distance 3 from $u_i$. Therefore,

$$\sigma(u_i) = 2(n - 1) + (n - 1) + 3[m - (n - 1)]$$

$$= 3(n - 1) + 3 \left[ \frac{n(n - 1)}{2} - (n - 1) \right].$$

Hence, $\sigma(u_i) = 3 \left[ \frac{n(n - 1)}{2} \right].$

Similarly, for every vertex $v_i \in V$ there are two vertices in $U$ at distance 1 and the remaining $(n - 2)$ vertices of $U$ at a distance 3.

Also, $(2n - 4)$ subdivision vertices are at distance 2 and $[(m - 1) - 2d(u) - 1]$ number of vertices are at distance 4.

$$\sigma(v_i) = 2 + 2(2n - 4) + 3(n - 2) + 4[(m - 1) - 2(d(u) - 1)]$$

$$= 7n - 12 + 4[(nC_2 - 1) - 2((n - 1) - 1)] = 2n^2 - 3n = n(2n - 3).$$

Therefore,

$$ABCS[S(K_n)] = \sum_{uv \in E(S(K_n))} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}}$$

$$= \sum_{uv \in E(S(K_n))} \sqrt{\frac{7n^2 - 9n - 4}{n^2(6n^2 - 15n + 9)}}.$$

Since there are $2m$ edges in $S(K_n)$, $ABCS[S(K_n)] = 2m \times \sqrt{\frac{7n^2 - 9n - 4}{n^2(6n^2 - 15n + 9)}}$. □

Example 1. From the figure in $S(K_4)$, $\sigma(v_i) = 18$, $i = 1, 2, 3, 4$. Let $s_j$, $j = 1, 2, 3, 4, 5, 6$ be the subdivision vertices, then $\sigma(s_j) = 20$. Then,

$$ABCS[S(K_4)] = \sum_{uv \in E(G)} \sqrt{\frac{18 + 20 - 2}{20 \times 18}} = 12 \times \sqrt{\frac{36}{360}} = 3.7947.$$
By the formula for $m = 6$ and $n = 4$,

$$ABCS[S(K_4)] = 2m \times \sqrt{\frac{7n^2 - 9n - 4}{n^2(6n^2 - 15m + 9)}} = 12 \times \sqrt{\frac{7(16) - 9(4) - 4}{4^2[6(16) - 15(4) + 9]}} = 3.7947.$$ 

**Theorem 5.2.** For a complete bipartite graph $K_{p,q}$ on $n$ vertices,

$$ABCS[S(K_{p,q})] = m \times \left[ \sqrt{\frac{7m + n + 4p - 10}{(3m + 4p - 4)(4m + n - 4)}} + \sqrt{\frac{7m + n + 4q - 10}{(3m + 4q - 4)(4m + n - 4)}} \right].$$ 

**Proof.** Partitioning the vertex set of subdivision graph of $K_{p,q}$ into three vertex set $U = \{u_1, u_2, u_3, \ldots, u_p\}$; $V = \{v_1, v_2, v_3, \ldots, v_q\}$; $W = \{w_1, w_2, w_3, \ldots, w_m\}$. Here $n = p + q$ and $m = pq$. For any edge in $S(K_{p,q})$, partitioning the edge set, $E = \{uv/u \in U \text{ or } V \text{ and } v \in W\}$. Let $E_1 = \{uv/u \in U \text{ and } v \in W\}$ and $E_2 = \{uv/u \in V \text{ and } v \in W\}$. Every vertex $u \in E_1$ is at a distance 1 from $q$ subdivision vertices, at a distance 2 from $q$ vertices of $V$, at a distance 4 from $(p-1)$ vertices of $U$, at a distance 3 from $(p-1)$ subdivision vertices and at a distance 3 from $(p-1)(q-1)$ subdivision vertices. Therefore,

$$\sigma(u) = q + 2q + 3(p-1) = 4(p-1) + 3(p-1)(q-1)$$

$$\sigma(u) = 3pq + 4p - 4 = 3m + 4p - 4.$$
Similarly, every vertex \( u \in E_2 \) is at a distance 1 from \( p \) subdivision vertices, at a distance 2 from \( p \) vertices of \( U \), at a distance 4 from \( (q - 1) \) vertices of \( U \), at a distance 3 from \( p(q - 1) \) subdivision vertices.

Therefore, \( \sigma(u) = p + 2p + 3p(q - 1) + 4(q - 1) \sigma(u) = 3pq + 4q - 4 = 3m + 4q - 4. \)

For every vertex \( v \in E_1 \) or \( E_2 \), two vertices are at a distance 1, \((p - 1)\) and \((q - 1)\) vertices of \( U \) and \( V \) are at a distance 3, \((p - 1)\) and \((q - 1)\) vertices are at distance 2 and \((p - 1)(q - 1)\) vertices at distance 4. Therefore, \( \sigma(v) = 2 + 3(p + q - 2) + 2[(p - 1) + (q - 1)] + 4(p - 1)(q - 1). \)

\( \sigma(v) = 4m + n - 4. \)

By the definition of Atom bond connectivity status index of a graph \( G \),

\[
ABCS[S(K_{p,q})] = \sum_{uv \in E_1} \sqrt{\frac{7m + n + 4p - 10}{(3pq + 4p - 4)(4m + n - 4)}} + \sum_{uv \in E_2} \sqrt{\frac{7m + n + 4q - 10}{(3pq + 4q - 4)(4m + n - 4)}}.
\]

Hence,

\[
ABCS[S(K_{p,q})] = m \times \left[ \sqrt{\frac{7m + n + 4p - 10}{(3m + 4p - 4)(4m + n - 4)}} + \sqrt{\frac{7m + n + 4q - 10}{(3m + 4q - 4)(4m + n - 4)}} \right].
\]

\( \square \)

**Example 2.** From the figure 2 in, \( S(K_{2,3}), \sigma(u_1) = \sigma(u_2) = 22, \sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 26. \) Let \( w_i, i = 1, 2, 3, 4, 5, 6 \) be the subdivision vertices. Then, \( \sigma(w_i) = \)
25 for \(i = 1, 2, 3, 4, 5, 6\). Now,

\[
ABCS[(K_{2,3})] = \sum_{uv \in E_1} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}} + \sum_{uv \in E_2} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}}
\]

\[
= \sum_{uv \in E_1} \sqrt{\frac{22 + 25 - 2}{22(25)}} + \sum_{uv \in E_2} \sqrt{\frac{26 + 25 - 2}{26(25)}} = 3.3635.
\]

By the Formula for \(m = 6, n = 5, p = 2, q = 3\),

\[
ABCS[(K_{2,3})] = m \times \left[ \sqrt{\frac{7m + n + 4p - 10}{(3m + 4p - 4)(4m + n - 4)}} + \sqrt{\frac{7m + n + 4q - 10}{(3m + 4q - 4)(4m + n - 4)}} \right]
\]

\[
= 6 \times \left[ \sqrt{\frac{7(6) + 5 + 4(2) - 10}{3(6) + 4(2) - 4}[4(6) + 5 - 4]} + \sqrt{\frac{7(6) + 5 + 4(3) - 10}{3(6) + 4(3) - 4}[4(6) + 5 - 4]} \right]
\]

\[
= 3.3635.
\]

**Theorem 5.3.** If \(P_n\) is a path graph on \(n\) vertices, then

\[
ABCS[(P_n)] = \sum_{i=1}^{2n-2} \sqrt{\frac{2n(2n-1) + i(i - 2n) + (i + 1)[(i + 1) - 2n] - 2}{[n(2n-1) + i(i - 2n)][n(2n-1) + (i + 1)[(i + 1) - 2n]]}}.
\]

**Proof.** The subdivision graph of \(P_n\) has \(n + n - 1 = 2n - 1\) vertices. Let \(v_1, v_2, v_3, \ldots, v_{2n-1}\) be the vertices, where \(v_i\) is adjacent to \(v_{i+1}, i = 1, 2, 3, \ldots, 2n - 2\). Therefore,

\[
\sigma(v_i) = \left(\frac{(2n-1)^2 + (2n-1)}{2} + i(i - (2n - 1) - 1) \right)
\]

\[
= n(2n - 1) + i(i - 2n)
\]

\[
\sigma(v_{i+1}) = n(2n - 1) + (i + 1)[(i + 1) - 2n]
\]

\[
[\sigma(u) + \sigma(v) - 2] = 2n(2n - 1) + i(i - 2n) + (i + 1)[(i + 1) - 2n] - 2
\]

\[
[\sigma(u) \cdot \sigma(v)] = [n(2n - 1) + i(i - 2n)][n(2n - 1) + (i + 1)[(i + 1) - 2n]].
\]

Hence,

\[
ABCS[(P_n)] = \sum_{i=1}^{2n-2} \sqrt{\frac{2n(2n-1) + i(i - 2n) + (i + 1)[(i + 1) - 2n] - 2}{[n(2n-1) + i(i - 2n)][n(2n-1) + (i + 1)[(i + 1) - 2n]]}}.
\]

\(\square\)
Example 3. From the figure in, \( S(P_4) \). If \( v_1, v_2, v_3 \) are the subdivision vertices then, \( \sigma(u_1) = 21, \sigma(v_1) = 16, \sigma(u_2) = 13, \sigma(v_2) = 12, \sigma(u_3) = 13, \sigma(v_3) = 16, \sigma(v_4) = 21 \), and also

\[
ABCS[S(P_4)] = \sum_{uv \in E[S(P_4)]} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}} = \sqrt{\frac{21 + 16 - 2}{(21)(16)}} + \sqrt{\frac{13 + 16 - 2}{(13)(16)}} + \sqrt{\frac{13 + 12 - 2}{(13)(12)}} + \sqrt{\frac{13 + 16 - 2}{(13)(16)}} + \sqrt{\frac{21 + 16 - 2}{(21)(16)}}
\]

\[= 0.3227 + 0.3602 + 0.3839 + 0.3839 + 0.3602 + 0.3227 = 2.1336.\]

By the formula given in Theorem 5.3

\[
ABCS[S(P_4)] = \sum_{i=1}^{6} \sqrt{\frac{56 + i(i-8) + (i+1)[(i+1)-8]-2}{[28 + i(i-8)][28 + (i+1)[(i+1)-8]]}} = 2.1336.
\]

Theorem 5.4. For a cycle \( C_n, n \geq 3 \) on \( n \) vertices,

\[
ABCS[S(C_n)] = \frac{2}{n} \left( \sqrt{2n^2 - 2} \right).
\]

Proof. The subdivision graph of \( C_n \) has \( 2n \) vertices. For any vertex \( u \) of \( S(C_n) \), \( \sigma(u) = 2 \left[ 1 + 2 + \cdots + \frac{n-1}{2} \right] + \frac{n}{2} = \frac{(2n)^2}{4} = n^2 \). Therefore, \( ABCS(C_n) = 2n \times \sqrt{\frac{2n^2 - 2}{n^4}} = \frac{2}{n} \left( \sqrt{2n^2 - 2} \right) \). \( \square \)
Example 4. Let \( v_i, i = 1, 2, 3, 4 \) be the subdivision vertices then from the above figure 4 in \( S(C_4) \), \( \sigma(u_i) = \sigma(v_i) = 16, i = 1, 2, 3, 4 \). Then, \( ABCS[S(P_4)] = \sum_{uv \in E[S(C_4)]} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}} = 8 \sqrt{\frac{16 + 16 - 2}{16(16)}} = 2.7386 \).

By the formula, \( ABCS[S(C_n)] = 2 \sqrt{\frac{2n^2 - 2}{2n^2 - 2 - d(u)}} = 2 \sqrt{\frac{\sqrt{32} - 2}{4}} = 2.7386 \).

6. Atom-bond connectivity status index of graphs formed by using the complete graph

In this section we have obtained the atom-bond connectivity status index of some graphs, which are defined in [1].

Proposition 6.1. For a complete graph \( K_n \) with \( n \geq 3 \), let \( e_i, i = 1, 2, \ldots, k \), \( 1 \leq k \leq n - 2 \), be the distinct edges all being incident with a single vertex. The graph \( K_{a_n}(k) \) is obtained by deleting \( e_i, i = 1, 2, \ldots, k \) from \( K_n \). Then,

\[
ABCS(K_{a_n}(k)) = [n - k - 1] \times \sqrt{\frac{2n + k - 4}{n(n - 1)}} + \left[ \frac{k(k - 1)}{2} \right] \times \sqrt{\frac{2n - 4}{(n - 1)^2}}.
\]

Proof. By the equality part of Theorem 3.1,

\[
ABCS(G) = \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [d(u) + d(v)]}{(2n - 2 - d(u))(2n - 2 - d(v))}}.
\]

The edge set \( E(K_{a_n}(k)) \) can be partitioned into four sets \( E_1, E_2, E_3 \) and \( E_4 \), where \( E_1 = \{uv/d(u) = n - 1 - k \text{ and } d(u) = n - 1\} \), \( E_2 = \{uv/d(u) = n - 2 \text{ and } d(u) = n - 2\} \), \( E_3 = \{uv/d(u) = n - 2 \text{ and } d(u) = n - 1\} \), \( E_4 = \{uv/d(u) = n - 1 \text{ and } d(u) = n - 1\} \), with \( |E_1| = n - k - 1 \), \( |E_2| = (k - 1)/2 \), \( |E_3| = (n - k - 1)k \), \( |E_4| = (n - k - 1)(n - k - 2)/2 \). Also \( \text{diam}(K_{a_n}(k)) = 2 \).
Therefore,

\[\text{ABCS} (K_{a_n}(k)) = \sum_{uv \in E(G)} \sqrt{\frac{4n - 6 - [n - 1 - k + n - 1]}{(2n - 2 - (n - 1 - k)) (2n - 2 - (n - 1))}}\]

\[+ \sum_{uv \in E_2(G)} \sqrt{\frac{4n - 6 - [n - 2 + n - 2]}{(2n - 2 - (n - 2)) (2n - 2 - (n - 2))}}\]

\[+ \sum_{uv \in E_3(G)} \sqrt{\frac{4n - 6 - [n - 2 + n - 1]}{(2n - 2 - (n - 2)) (2n - 2 - (n - 1))}}\]

\[+ \sum_{uv \in E_4(G)} \sqrt{\frac{4n - 6 - [n - 1 + n - 1]}{(2n - 2 - (n - 1)) (2n - 2 - (n - 1))}}.\]

Therefore

\[\text{ABCS}(K_{a_n}(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n + k - 4}{n(n-1)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n - 2}{n^2}}\]

\[+ \sum_{uv \in E_3(G)} \sqrt{\frac{2n - 3}{n(n-1)}} + \sum_{uv \in E_4(G)} \sqrt{\frac{2n - 4}{(n-1)^2}}.\]

Hence,

\[\text{ABCS}(K_{a_n}(k)) = [n - k - 1] \times \sqrt{\frac{2n + k - 4}{n(n-1)}} + \left[ \frac{k(k-1)}{2} \right]\]

\[\times \sqrt{\frac{2n - 2}{n^2}} + [(n - k - 1)k] \times \sqrt{\frac{2n - 3}{n(n-1)}}\]

\[+ \left[ \frac{(n - k - 1)(n - k - 2)}{2} \right] \times \sqrt{\frac{2n - 4}{(n-1)^2}}.\]

\[\square\]

**Proposition 6.2.** For a complete graph \(K_n\) with \(n \geq 3\), let \(f_i, i = 1, 2, \ldots, k, 1 \leq k \leq \lfloor n/2 \rfloor\), be independent edges. The graph \(K_{b_n}(k)\) is obtained by deleting \(f_i,\)
Therefore, $i = 1, 2, \ldots, k$ edges from $K_n$. Then,

$$ABCS(Kb_n(k)) = [2k(n-2k)] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[ \frac{(n-2k)(n-2k-1)}{2} \right] \times \sqrt{\frac{2n-4}{(n-1)^2}} + \left( \frac{2k(2k-1)}{2} \right) - k \times \sqrt{\frac{2n-2}{n^2}}.$$ 

Proof. The edge set $E(Kb_n(k))$ can be partitioned into three sets $E_1$, $E_2$ and $E_3$, where $E_1 = \{uv/d(u) = n-2$ and $d(v) = n-1\}$, $E_2 = \{uv/d(u) = n-1$ and $d(v) = n-1\}$, $E_3 = \{uv/ d(u) = n-2$ and $d(v) = n-2\}$. It is easy to check that $|E_1| = 2k(n-2k)$, $|E_2| = ((n-2k)(n-2k-1)/2)$ and $|E_3| = (2k(2k-1)/2) - k$.

Also $diam((Kb_n(k)) = 2$.

By the equality part of Theorem 3.1,

$$ABCS(G) = \sum_{uv \in E(G)} \sqrt{\frac{4n-6 - [d(u) + d(v)]}{(2n-2 - d(u))(2n-2 - d(v))}}$$

$$ABCS(Kb_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{4n-6 - [n-2 + n-1]}{(2n-2 - (n-2))(2n-2 - (n-1))}}$$

$$+ \sum_{uv \in E_2(G)} \sqrt{\frac{4n-6 - [n-1 + n-1]}{(2n-2 - (n-1))(2n-2 - (n-1))}}$$

$$+ \sum_{uv \in E_3(G)} \sqrt{\frac{4n-6 - [n-2 + n-2]}{(2n-2 - (n-2))(2n-2 - (n-2))}}.$$ 

Therefore,

$$ABCS(Kb_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n-3}{n(n-1)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n-4}{(n-1)^2}} + \sum_{uv \in E_3(G)} \sqrt{\frac{2n-2}{n^2}}.$$ 

Hence,

$$ABCS(Kb_n(k)) = [2k(n-2k)] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[ \frac{(n-2k)(n-2k-1)}{2} \right] \times \sqrt{\frac{2n-4}{(n-1)^2}} + \left( \frac{2k(2k-1)}{2} \right) - k \times \sqrt{\frac{2n-2}{n^2}}.$$ 

□
Proposition 6.3. For a complete graph $K_n$, $n \geq 3$, let $V_k$ be a $k$-element subset of the vertex set $2 \leq k \leq n-1$. The graph $K_{c_n}(k)$ is obtained by deleting from all the edges connecting pairs of vertices from $V_k$. Then,

$$ABCS(K_{c_n}(k)) = [(n-k)k] \times \sqrt{\frac{2n+k-5}{(n-2+k)(n-1)}} + \left(\frac{(n-k)(n-k-1)}{2}\right) \times \sqrt{\frac{2n-4}{(n-1)^2}}.$$ 

Proof. The edge set $E(K_{c_n}(k))$ can be partitioned into two sets $E_1$ and $E_2$, where $E_1 = \{uv/d(u) = n-k \text{ and } d(v) = n-1\}$ and $E_2 = \{uv/d(u) = n-1 \text{ and } d(v) = n-1\}$. Also $|E_1| = (n-k)k$, $|E_2| = (n-k)(n-k-1)/2$. and $\text{diam}(K_{b_n}(k)) = 2$.

By the equality part of Theorem 3.1,

$$ABCS(K_{c_n}(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{4n-6-[n-k+n-1]}{(2n-2-(n-k))(2n-2-(n-1))}} + \sum_{uv \in E_2(G)} \sqrt{\frac{4n-6-[n-1+n-1]}{(2n-2-(n-1))(2n-2-(n-1))}}.$$ 

Therefore,

$$ABCS(K_{c_n}(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n+k-5}{(n-2+k)(n-1)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n-4}{(n-1)^2}}.$$ 

Hence the result follows. \qed

Proposition 6.4. For a complete graph $K_n$ with $n \geq 5$, let $3 \leq k \leq n$. The graph $K_{d_n}(k)$ is obtained by deleting from $K_n$, the edges belonging to a $k$-membered cycle. Then

$$ABCS(K_{d_n}(k)) = \left[\frac{k(k-1)}{2} - k\right] \times \sqrt{\frac{2n}{(n+1)^2}} + [(n-k)k] \times \sqrt{\frac{2n-2}{(n-1)(n-1)}} + \left(\frac{(n-k)(n-k-1)}{2} - k\right) \times \sqrt{\frac{2n-4}{(n-1)^2}}.$$ 

Proof. The edge set $E(K_{d_n}(k))$ can be partitioned into three sets $E_1$, $E_2$ and $E_3$, where $E_1 = \{uv/d(u) = n-3 = d(v)\}$, $E_2 = \{uv/d(u) = n-3 \text{ and } d(v) = n-1\}$, $E_3$
\[ uv / d(u) = n - 1 = d(v) \]. It is easy to check that and \(|E_1| = (k(k-1)/2) - k, |E_2| = (n-k)k \] and \(|E_3| = ((n-k)(n-k-1)/2). Also \(\text{diam}(Kd_n(k)) = 2\).

By the equality part of Theorem 3.1,

\[
ABCS(Kd_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n}{(n+1)^2}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n-2}{(n+1)(n-1)}} + \sum_{uv \in E_3(G)} \sqrt{\frac{2n-4}{(n-1)^2}}.
\]

Hence the result follows. \(\square\)

7. Conclusion

In this paper we have obtained bounds for the atom-bond connectivity status index of graph in terms of degree and diameter. Gave alternate proof of atom-bond connectivity status index of some standard graphs. Obtained atom-bond connectivity status index of subdivision graph of some graphs and edge deleted graph obtained from complete graph.

References


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