RECURRENCE RELATION UNDER EFROS THEOREM

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ABSTRACT. In the present manuscript, we apply Efros theorem to establish certain recurrence relation. The established results supposed to be new and general. By giving particular values to the parameters, a number of new and known results can be established.

1. INTRODUCTION

First, we will give a brief account of the Efros theorem, Laplace transform [1] and Parseval Goldstein theorem [4], which will be used to derive our main theorem.

(a) The Efros theorem [5] states that if G(p) and q(p) are two analytic function given by:

\[ F(p) = L[f(t)], \]
\[ G(p)e^{-\tau q(p)} = L[g(t, \tau)] , \]

then

\[ G(p)F(q(p)) = L \left[ \int_{0}^{\infty} f(\tau)g(t, \tau)d\tau \right] . \]
(b) Laplace transform can be defined as follows
\[ f(p) = L[f(t); p] = f(p) = \int_0^\infty e^{-pt} f(t) dt, \]

(c) Parseval-Goldstein theorem states that if \( \phi_1(p) = L[h_1(t)] \) and \( \phi_2(p) = L[h_2(t)] \), then
\[ \int_0^\infty \phi_1(t) h_2(t) dt = \int_0^\infty \phi_2(t) h_1(t) dt. \]

2. Main Result

**Theorem 2.1.** If \( \lambda > n - 1, R(\sigma + \lambda + 1) > 0, (p + a) > 0, \)
\[ F(p) = L[f(t)], \]
and
\[ G(p)e^{-\tau q(p)} = L[g(t, \tau)], \]
then
\[ L\left[ t^n (t + a)^{-\lambda - 1} G(t)e^{-\tau q(t)}; p \right] = \sum_{r=0}^{n} \left( -1 \right)^{n-r} \frac{a^{n-r}}{\Gamma(\lambda - r + 1)} \frac{n!}{C_r b^{n-r} t^{\lambda-r}}. \]

where \( f(t) = o(t) \) for some \( t \) and \( f(t) = O(e^{-ctt^\mu}) \) for large \( t \).

**Proof.** Since from (2), p. 127, we have
\[ g(r) = \int_0^\infty e^{-pt} f(t) dt, \]
\[ g(r) = \int_0^\infty e^{-pt} t^\lambda dt. \]

Now \( L\left[ t^n \left( t + a \right)^{-\lambda - 1} G(t)e^{-\tau q(t)}; p \right] = \sum_{r=0}^{n} \left( -1 \right)^{n-r} \frac{a^{n-r}}{\Gamma(\lambda - r + 1)} \frac{n!}{C_r b^{n-r} t^{\lambda-r}}. \)

Further, by virtue of Leibnitz theorem we have
\[ \frac{d^n}{dt^n} (t^\lambda e^{-bt}) = e^{-bt} \sum_{r=0}^{n} \left( -1 \right)^{n-r} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - r + 1)} \Gamma(\lambda - r + 1) \frac{n!}{C_r b^{n-r} t^{\lambda-r}}. \]

Therefore, if we take \( f(t) = t^\lambda e^{-bt}, p^n L[f(t); p] = L[f^n(t); p] \), where \( f(0) = f'(0) = f''(0) \ldots f^{n-1}(0) \) and \( f^n(t) \) stands for \( \frac{d^n}{dt^n} [f(t)] \), then
\[ \Gamma(\lambda + 1) p^n (p + b)^{-\lambda - 1} = L\left[ e^{-bt} \sum_{r=0}^{n} \left( -1 \right)^{n-r} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - r + 1)} \Gamma(\lambda - r + 1) \frac{n!}{C_r b^{n-r} t^{\lambda-r}}; p \right]. \]
Using Parseval Goldstein theorem [4] in the above equation (1.1) and (2.1), we get
\[
\int_0^\infty e^{-at}t^n(t + b)^{-\lambda - 1}G(t)e^{-\tau(q(t))}dt = \sum_{r=0}^n \frac{(-1)^{n-r}b^{n-r}}{\Gamma(\lambda - r + 1)}\int_0^\infty x^{\lambda - r}e^{-bx}g(x, \tau)dx.
\]
Replacing \( b \) as \( a \) and \( a \) as \( p \) then we get
\[
\int_0^\infty e^{-at}t^n(t + a)^{-\lambda - 1}G(t)e^{-\tau(q(t))}dt = \sum_{r=0}^n \frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda - r + 1)}\int_0^\infty x^{\lambda - r}e^{-bx}g(x, \tau)dx.
\]
(2.2)

which is supposed to be new result.

Taking \( \tau = 0 \) in the above equation (2.2), then we get
\[
(2.3) \quad L[t^n(t + a)^{-\lambda - 1}G(t); p] = \sum_{r=0}^n \frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda - r + 1)}\int_0^\infty x^{\lambda - r}e^{-bx}g(x, \tau)dx.
\]

3. Example

If we take \( f(t) = t \), then ( [2], p. 137), we have \( L[t^\nu; p] = \Gamma(\nu + 1)p^{-\nu - 1}, R(\nu) > -1 \) and \( R(p) > 0 \). Substituting this value in the above equation (2.3), we get
\[
L[t^{n+\nu}(t + a)^{-\lambda - 1}; p] = \Gamma(\nu + 1)\sum_{r=0}^n \frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda - r + 1)}\int_0^\infty e^{-ax}x^{\lambda - r}p^{-\nu - 1}dx.
\]

Solving the right-hand side with the help of the result ( [2], p. 129), we get
\[
L[t^{\lambda - 1}(t + a)^{-\nu}; p] = \frac{p^{-\lambda}a^{-\nu}}{\Gamma(\nu)}E[\lambda; \nu :: ap],
\]
where \( R(\lambda) > 0, R(p) > 0, \)
\[
\sum_{r=0}^n \frac{(-1)^{n-r}(ap)^n}{\Gamma(\lambda - r + 1)}nC_r E[\lambda - r + 1, \nu + 1 :: ap]
\]
(3.1)

= \frac{1}{\Gamma(\lambda + 1)}E[n + \nu + 1, \lambda + 1 :: ap].
Now, with the help of the result [7]:

\[ E[\mu, \lambda :: x] = \Gamma(\mu)\Gamma(\lambda)e^{x^2}x^{-\frac{1}{2}(1-\mu-\lambda)}W^{(x)}_{\left(\frac{1-\mu-\lambda}{2}\right)} \]

(3.1) can be written as follows

\[ \sum_{r=0}^{n} (-1)^{n-r} nC_r x^{\frac{n-r}{2}}W_{\left(k+\frac{r}{2}, m+\frac{r}{2}\right)} = \frac{\sqrt{m+n-k+1}}{\sqrt{m-k+\frac{1}{2}}} W_{\left(k-r, m+\frac{r}{2}\right)} \]

On taking \( n = 1 \), (3.2) gives rise to result ( [6], p. 27)

\[ \left(m - k + \frac{1}{2}\right) W_{\left(k-\frac{1}{2}, m+\frac{1}{2}\right)} + x^\frac{1}{2}W_{k,m} = W_{\left(k+\frac{1}{2}, m+\frac{1}{2}\right)} \]

Now, substituting in the above result (3.2) \( x = \frac{1}{2}y^2 \) and multiplying both side by \( e^{\frac{3}{4}y^2}y^{\nu-2m-n-1} \), we get

\[ \sum_{r=0}^{n} (-1)^{n-r} nC_r \frac{e^{\frac{3}{4}y^2}y^{\nu-2m-n-\frac{1}{2}}}{2^{2}2^{-\frac{1}{2}}} W_{\left(k+\frac{r}{2}, m+\frac{r}{2}\right)} \]

(3.3)

Taking images in Hankel transform ( [3]; p. 84) of both the side of (3.3) and substituting \( k + \frac{3m}{4} - \frac{\nu}{4} - \frac{1}{4}, k + \frac{m}{2} + \frac{\nu}{2} + \frac{1}{4} \) and \( \frac{\nu^2}{2} \) as \( k, m \) and \( x \) respectively, we get

\[ \sum_{r=0}^{n} \frac{\Gamma\left(m-n-k+\frac{1}{2}\right)}{\Gamma\left(m-k+n+\frac{1}{2}\right)} (-1)^{n-r} nC_r W_{\left(k-r, m\right)} = x^{\frac{1}{2}}W_{\left(k+r, m+\frac{1}{2}\right)} \]

(3.4)

On taking \( n = 1 \), (3.4) gives rise to result ( [6], p. 27).

REFERENCES


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