COVID-19 ANALYSIS BY SEIR MODEL WITH MULTIPLE CONTROLS

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Abstract. A SEIR mathematical model with multiple controls self-prevention, treatment and vaccination is formulated. The properties of Pontryagin’s maximum principle were verified and found the optimal levels of controls. Numerical simulations were shown to exhibit the flow of variables with or without control strategies.

1. Introduction

The control measures were followed to reduce the spread of the infectious diseases. As we all know ‘Prevention is better than cure’, it is better to be self preventive which is the primary control measure. The self prevention such as social distancing, washing hands often, using clean cloth while coughing/sneezing and self quarantine helps us to prevent the spread of the disease. The medical care helps to cure the disease and makes to feel better. So treatment plays an important role as a control measure. Vaccination is another key control measure which develops an immune in our body and control the infection of the diseases.

Mathematical modeling analysis explains the transmission process and control of the infectious diseases. Optimal control theory for infectious diseases to used

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The first COVID - 19 case was reported on 1 April 2020 in Vellore district of Tamilnadu, India. In this paper, the data of COVID - 19 infection in Vellore district has been collected from the Vellore Collectorate press release. We have formulated and analysed a SEIR model for COVID - 19 with multiple controls self - prevention, treatment and vaccination.

2. FORMULATION OF MATHEMATICAL MODEL AND CONTROL SET

The SEIR model with multiple controls for COVID-19 is represented by ordinary differential equations as follows:

\[
\begin{align*}
\frac{dS}{dt} &= \omega N(t) - \beta S(t)I(t) - \mu S(t) - (u_1 + u_3)S(t) \\
\frac{dE}{dt} &= \beta S(t)I(t) - (\mu + \alpha)E(t) - (u_1 + u_2)E(t) \\
\frac{dI}{dt} &= \alpha E(t) - (\gamma + \mu + d + u_2)I(t) \\
\frac{dR}{dt} &= (\gamma + u_2)I(t) + (u_1 + u_3)S(t) + (u_1 + u_2)E(t) - \mu R(t)
\end{align*}
\]

where \( S(t), E(t), I(t), R(t), N(t) \) are the Susceptible, Exposed, Infectious, Recovery state, Total population respectively, \( \omega \) - Average birth rate, \( \mu \) - Average death rate, \( \beta \) - Transition infectious rate, \( \gamma \) - Recovery rate and \( d \) - disease induced death rate. Here we assume \( u_1 \) - measures for self - prevention, \( u_2 \) - treatment measures and \( u_3 \) - vaccination as the controls for the model.

The transmission of SEIR model for COVID-19 with multiple controls is diagrammatically represented as follows.
For this system of equations, we define an objective functional similar as in [6]

\[
J = \min_{u_1, u_2, u_3 \in U} AI(t) + \frac{1}{2} \int_0^T \left( w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2 \right) dt
\]

which is a free problem system subject to infected.

Here \( AI(t) \) is the number of individuals, \( w_1, w_2, w_3 \) are the weight parameters for the cost of awareness for self-prevention, treatment and vaccination respectively.

The control set is defined as

\[
U = \{(u_1(t), u_2(t), u_3(t)) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1, 0 \leq u_3 \leq 1, t \in [0, 1]\}.
\]

3. Properties of Pontryagin’s maximum principle

Pontryagin’s maximum principle is a powerful method to compute the optimal controls. To use this, we have to prove the properties [3]:

\( P_1 \): The set of controls and the corresponding state variables is non empty.

\( P_2 \): The control set \( U \) is convex and closed.

\( P_3 \): The R.H.S. of the state system is bounded by a linear function in the state and control variables.

\( P_4 \): The integrand of the objective functional is convex on \( U \) and is bounded below by \(-k_2 + k_1 |U|^\eta\), with \( k_1 > 0, k_2 > 0, \eta > 1\).

To prove this we need the following theorem and lemma.
Theorem 3.1. Existence and Uniqueness Theorem Let \( \frac{du}{dx} = f(x, y), y(x_0) = y_0 \) and \( f(x, y) \) be continuous on a domain \( \{D = (x, y)/x_0 - a < x < x_0 + a; y_0 - b < y < y_0 + b\} \) with the Lipschitz condition \( ||f(x, y_1) - f(x, y_2)|| \leq K||y_1 - y_2|| \) where \( K \) is a positive integer, then there exist bounded solution in \( D \). Further more, if \( f(x, y) \) is Lipschitz continuous with respect to \( y \) on a rectangle \( R = \{(x, y)/x_0 - c < x < x_0 + c; y_0 - b < y < y_0 + b; c < a\} \) then there is a unique solution \( y(t) \) in \( R \).

Lemma 3.1. If \( f(x, y) \) has continuous partial derivative \( \frac{\partial f}{\partial y_1} \) on a bounded closed domain \( R \), is a set of real numbers. Then it satisfies a Lipschitz condition in \( R \).

To prove \( P_1 \): From (2.1),

\[
(3.1) \quad F_1 = \omega N(t) - \beta S(t) I(t) - \mu S(t) - (u_1 + u_3) S(t)
\]

\[
(3.2) \quad F_2 = \beta S(t) I(t) - (\mu + \alpha) E(t) - (u_1 + u_2) E(t)
\]

\[
(3.3) \quad F_3 = \alpha E(t) - (\gamma + \mu + d + u_2) I(t)
\]

\[
(3.4) \quad F_4 = (\gamma + u_2) I(t) + (u_1 + u_3) S(t) + (u_1 + u_2) E(t) - \mu R(t)
\]

From (3.1), (3.2), (3.3) and (3.4),

\[
\left| \frac{\partial F_1}{\partial S} \right| < \infty, \quad \left| \frac{\partial F_1}{\partial E} \right| < \infty, \quad \left| \frac{\partial F_1}{\partial I} \right| < \infty, \quad \left| \frac{\partial F_1}{\partial R} \right| < \infty
\]

\[
\left| \frac{\partial F_2}{\partial S} \right| < \infty, \quad \left| \frac{\partial F_2}{\partial E} \right| < \infty, \quad \left| \frac{\partial F_2}{\partial I} \right| < \infty, \quad \left| \frac{\partial F_2}{\partial R} \right| < \infty
\]

\[
\left| \frac{\partial F_3}{\partial S} \right| < \infty, \quad \left| \frac{\partial F_3}{\partial E} \right| < \infty, \quad \left| \frac{\partial F_3}{\partial I} \right| < \infty, \quad \left| \frac{\partial F_3}{\partial R} \right| < \infty
\]

\[
\left| \frac{\partial F_4}{\partial S} \right| < \infty, \quad \left| \frac{\partial F_4}{\partial E} \right| < \infty, \quad \left| \frac{\partial F_4}{\partial I} \right| < \infty, \quad \left| \frac{\partial F_4}{\partial R} \right| < \infty
\]

Here the partial derivatives are continuous and bounded. Hence there exist a unique solution of (2.1).

To prove \( P_2 \): \( U \) is closed, by the definition. Let us take two controls \( u_2, u_3 \in U \) and \( \alpha \in [0, 1] \Rightarrow \alpha u_2 + (1 - \alpha) u_3 \geq 0 \). As \( u_2 \leq 1 \Rightarrow \alpha u_2 \leq \alpha \) and \( u_3 \leq 1 \Rightarrow (1 - \alpha) u_3 \leq (1 - \alpha) \).

\[
\therefore \ 0 \leq \alpha u_2 + (1 - \alpha) u_3 \leq 1, \forall u_2, u_3 \in U. \quad \text{Hence } U \text{ is convex, } P_2 \text{ is satisfied}
\]

To prove \( P_3 \): \( F_1 \leq \omega N - (u_1 + u_3) S \).
From (3.2), \( F_2 \leq \beta SI - (u_1 + u_2) \).
From (3.3), \( F_3 \leq \alpha E - u_2 I \).
From (3.4), \( F_4 \leq (\gamma + u_2)I + (u_1 + u_3)S + (u_1 + u_2)E \).

The state system can be rewritten in the matrix form,

\[
\vec{F}(t, \vec{X}, U) \leq \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \beta S & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \gamma & 0 \\
\end{bmatrix} \cdot \begin{bmatrix}
S \\
E \\
I \\
R \\
\end{bmatrix} + \begin{bmatrix}
-S \\
-E \\
-I \\
S + E + I \\
\end{bmatrix} \cdot \begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t) \\
\end{bmatrix}
\]

which can be written as the linear combination of controls,

\[
|\vec{G}(t, \vec{X}, U)| \leq \|\vec{m}\| |\vec{X}| + \|\vec{S} + \vec{E} + \vec{I}\| |u_1(t), u_2(t), u_3(t)| \leq K_2 \left[|\vec{X}| + |u_1(t), u_2(t), u_3(t)|\right].
\]

As \( \vec{S}, \vec{E}, \vec{I} \), are bounded and \( K_2 \) is an upper bound. Hence R.H.S. is bounded by the sum of state and control variables, \( P_3 \) is satisfied.

**To prove** \( P_4 \): The control and state variables are non-negative and the control variables \( u_1, u_2, u_3 \in U \) is also convex and closed. Then the integrand of the objective functional (2.2) is convex on the set \( U \). Now,

\[
J = AI(t) + \frac{w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2}{2} \geq -AI(t) + \frac{w}{2}(u_1^2 + u_2^2 + u_3^2) = -K_2 + K_1(u_1, u_2, u_3)^2,
\]

where \( w = w_1 + w_2 + w_3 \) and \( K_1 = \frac{w}{2} \).

Here \( K_2 > 0 \) as it depends on \( I \), \( K_1 > 0 \) and \( \eta = 2 > 1 \). Hence \( P_4 \) is satisfied.

\( \therefore \) The control system satisfies the properties of Pontryagin’s maximum principle then there exists an optimal control which we construct in the next section.

4. **Existence of optimal control**

The properties for Pontryagin’s maximum principle are satisfied, which converts (2.1) into a problem of minimizing.
An Hamiltonian $H$ is defined as
\[
H(S,E,I,R) = AI(t) + \frac{w_1 u_1^2}{2} + \frac{w_2 u_2^2}{2} + \frac{w_3 u_3^2}{2} + \lambda_S(\omega N - \beta SI - \mu S - (u_1 + u_3) S) \\
+ \lambda_E(\beta SI - (\mu + \alpha) E - (u_1 + u_2) E) \\
+ \lambda_I(\alpha E - (\gamma + \mu + d + u_2) I) \\
+ \lambda_R((\gamma + u_2) I + (u_1 + u_3) S + (u_1 + u_3) E - \mu R).
\]

The adjoint system is given by
\[
\frac{d\lambda_S}{dt} = \lambda_S(\beta I + \mu + u_1 + u_3) - \lambda_E\beta I - (u_1 + u_3)\lambda_R \\
\frac{d\lambda_E}{dt} = \lambda_E(\mu + \alpha + u_1 + u_2) - \alpha\lambda_I - \lambda_R(u_1 + u_2) \\
\frac{d\lambda_I}{dt} = \lambda_S\beta S - \lambda_E\beta S + \lambda_I(\gamma + \mu + d + u_2) - (\gamma + u_2)\lambda_R - A \\
\frac{d\lambda_R}{dt} = \mu\lambda_R
\]
with the final conditions $\lambda_S(T) = \lambda_E(T) = \lambda_I(T) = \lambda_R(T) = 0$ for free problem.

The optimal controls can be obtained from the optimality conditions, $\frac{\partial H}{\partial u_1} = 0; \frac{\partial H}{\partial u_2} = 0; \frac{\partial H}{\partial u_3} = 0$ we have
\[
u^*_1 = \frac{S(\lambda_S - \lambda_R) + E(\lambda_E - \lambda_R)}{w_1}, \\
u^*_2 = \frac{E(\lambda_E - \lambda_R) + I(\lambda_I - \lambda_R)}{w_2}, \\
u^*_3 = \frac{S(\lambda_S - \lambda_R)}{w_3}.
\]
Hence the optimal controls are
\[
u^*_1 = \min\left(1, \max\left(0, \frac{S(\lambda_S - \lambda_R) + E(\lambda_E - \lambda_R)}{w_1}\right)\right), \\
u^*_2 = \min\left(1, \max\left(0, \frac{E(\lambda_E - \lambda_R) + I(\lambda_I - \lambda_R)}{w_2}\right)\right), \\
u^*_3 = \min\left(1, \max\left(0, \frac{S(\lambda_S - \lambda_R)}{w_3}\right)\right).
\]
5. Numerical Analysis

The birth rate is $\omega = 0.0477$ and death rate is $\mu = 0.0203$ for Vellore district and the transmission rate is $\beta = 0.011$, the recovery rate is $\gamma = 0.96$ and the death rate is $d = 0.017$ for COVID-19.

Figure (2) represents the Confirmed cases, Recovered cases and Death cases of Covid-19 in Vellore district as on 30.11.2020. Figure (3) shows the flow of Susceptible, Exposed, Infectious and Recovery for the model.
Flow of the variables with and without controls

Figure 4. Susceptible and Exposed class

Figure 5. Infectious and Recovered class

Figure (4) (left) shows that the Susceptible with all the controls is less than the Susceptible without controls, also Susceptible with vaccination is less than Susceptible with self prevention and treatment and figure (4) (right) shows that the Exposed with all the controls is less than the Exposed without controls, also the Exposed with vaccination is higher than the Exposed with Self-Prevention and treatment.

It is clear from figure (5) (left) that the Infectious with all the controls is less than the Infectious without controls, also the Infectious with vaccination is higher than the Infectious with Self-Prevention and treatment and from figure (5) (right) that the recovery with Self-Prevention and treatment is higher than the recovery without controls, the recovery with vaccination and the recovery with all controls.
6. CONCLUSION

A SEIR model with multiple control measures self - Prevention, treatment and vaccination was formulated. Properties of Pontryagin’s maximum principle were verified. By Pontryagin’s maximum principle, the optimal levels of controls were analysed. Transition of COVID-19 in Vellore district and the flow of variables with controls with respect to time were exhibited in numerical simulations. The graphs in the Susceptible, Exposed, Infectious and recovery class with and without controls shows Vaccination can reduce the Susceptible, Exposed and Infectious can be controlled with all the controls and Recovered can be increased by Self-Prevention and treatment.

REFERENCES


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