AN ASSOCIATION BETWEEN GOLDEN RATIO AND AN INFINITE KOLAKOSKI SEQUENCE

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ABSTRACT. Fibonacci numbers ($F_n$) occur in nature and have many interesting properties also have many applications. The Kolakosi ($K$) roots ($\alpha, \beta$) are derived over the binary alphabet $\Sigma=\{1,2\}$ also described it on explicit formula for $F_n$. And the Kolakoski string of $i$ blocks and $j$ positions are strictly less than to Golden ratio ($\phi$) for every $Z^+$ is proved. The ratio of Kolakoski sequence is lies between 1 and $\phi$ is shown. Also the decimal fraction of Kolakoski numbers is shown for some positive integers.

1. INTRODUCTION

The Kolakosi sequence is an infinite sequence of symbols $\{1,2\}$ also known as the Olderburger Kolakoski sequence who has revealed his research in a paper by Rufus Olderburger in 1939, see [1]. It was recreated by William Kolakoski who discussed about it in 1965. An infinite Kolakoski sequence defined by its run-lengths, $K=1221121221221121122121$.... The Kolakoski ($K$) strings are generated over the binary alphabet $\Sigma=\{1,2\}$ and the roots ($\alpha, \beta$) are derived form $(3 \times 3)$ array also applied it on explicit formula for $F_n$, see [5,6]. And the relation between the Kolakoski sequence and Fibonacci sequence are discussed.

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Also the ratio of Kolakoski blocks $i$ and positions $j$ are established. For every $Z^+$
the Kolakoski strings $(i, j)$ is strictly less than to $\phi$ is proved. Further the ratio of
Kolakoski string is lies between 1 and $\phi$ is shown for every positive integer.

2. Basic Definitions and Preliminaries

**Definition 2.1.** A nonempty set $\Sigma$ of symbols, called the alphabet. The strings
are finite sequence of symbols from the alphabet. If the alphabet $\Sigma = \{1, 2\}$, then
$w = abab$ and $w = aaabbb$ are strings on $\Sigma$.

**Definition 2.2.** An infinite Kolakoski sequence $K$ has its own run length over the
binary alphabet $\Sigma = \{1, 2\}$ under two iterating operations,

$$
\sigma_0 \text{ (Even)} = \begin{cases} 
1 & \rightarrow 1 \\
2 & \rightarrow 11
\end{cases} \\
\sigma_1 \text{ (Odd)} = \begin{cases} 
1 & \rightarrow 2 \\
2 & \rightarrow 22
\end{cases}
$$

An infinite $K$ starts with the seed value as 1 under two iterating operations $\sigma_0$ and
$\sigma_1$ hence the classical Kolakoski sequence is

$$K = 12211221221121121122121122112211211221221121211211212$$

**Definition 2.3.** A palindrome is a string of numbers or letters that is the same as
forward as backward. For an example, the Kolakoski string $w^p_{(i,j)} = \{121, 212\}$ and
$w = abba$ are palindrome.

**Definition 2.4.** The Fibonacci numbers commonly denoted by $F_n$ which form a
sequence called the Fibonacci sequence, such that each number is the sum of the
two preceding ones, starting from 0 and 1. That is $F_0 = 0, F_1 = 1$ and $F_n = F_{(n-1)} + F_{(n-2)}$ for $n > 1$.

**Definition 2.5.** The irrational number $\frac{1+\sqrt{5}}{2}$ also known as the Golden ratio
is denoted by $\phi$ which is approximately equal to 1.618.

3. An Explicit Approach of $K_n$ on Golden Ratio

The recursive relation of the Fibonacci numbers is $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$,
$F_1 = 1, F_2 = 1$, see [2]. If the roots of the recursive relation $\alpha^2 = \alpha + 1$ be $\frac{1+\sqrt{5}}{2}$
and $\frac{1-\sqrt{5}}{2}$ then the value of the constant $\alpha$ is 1.618 is known as the Golden ratio,
see [3,4]. If $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$ then $F_3 = F_2 + F_1, F_4 = F_3 + F_2, F_5 = F_4 + F_3,$
$F_6 = F_5 + F_4, F_7 = F_6 + F_5$ and so on. An explicit formula for Fibonacci
numbers is $\frac{a^n-b^n}{a-\beta}$, see \[11\]. If $x^2 = x + 1$ then roots of the Fibonacci numbers is $\frac{1+\sqrt{5}}{2}$ hence an explicit formula for $F_n$ is $\frac{a^n-b^n}{a-\beta}$. If $n \geq 3$, $F_{n-1} = \frac{a^{n-1}-b^{n-1}}{a-\beta}$ and $F_{n-2} = \frac{a^{n-2}-b^{n-2}}{a-\beta}$. The solution of Kolakoski string of length $|w| = 3$ in $(3 \times 3)$ under $\sigma_0$ and $\sigma_1$ is $\alpha^2 = 5\alpha - 3$ and an explicit formula for $K_n$ is $K_{n-1} + K_{n-2} = K_n$, $\forall n \geq 2$ where $K_n = \frac{a^n-b^n}{a_2-a_3}$, $\forall n \geq 2$ and if $\alpha^n = \frac{a^{n-1}-a^{-1}}{a_2-a_3}, \beta^n = \frac{a^{n-2}-a^{n-2}}{a_2-a_3}$, $\forall n \geq 2$ then $K_n = \frac{a^{n-1}-a^{-1}+a^{n-2}}{a_2-a_3}$, $\forall n \geq 2$. Hence the roots $(\alpha, \beta)$ of $K_n$ be $\frac{5+\sqrt{13}}{2}$ and $\frac{5-\sqrt{13}}{2}$, see \[5,7,8\].

**Theorem 3.1.** If $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}, n \geq 3$ then $K_n = \frac{(pN_{12}^2-F_5\beta)^{(1+\alpha)}}{a-\beta}$.

**Proof.** Let, $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}, n \geq 3$ where $F_n$ is called $n^{th}$ Fibonacci number. And the root of the equation $x^2 = x + 1$ be $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$. An infinite $K$ of length $|w| = 3$ can be obtained in $(3 \times 3)$ array. Hence the roots of the equation will be $\gamma = -1$, $\beta = \frac{5+\sqrt{13}}{2}$, $\alpha = \frac{5-\sqrt{13}}{2}$. And $\alpha + \beta = 5$, $\alpha - \beta = \sqrt{13}$ or $\sqrt{13}$, $\alpha = 3$, $\alpha = 22 - 5\beta$ and $\beta = 22 - 5\alpha$. An explicit formula for $F_n$ is $F_n = \frac{a^n-b^n}{a-\beta}$. If $\alpha = \frac{5-\sqrt{13}}{2}$, then $\alpha - 1 = 2\alpha^{n-3}$. Let $\alpha_{n-1} = (pN_{12}^2 - F_5\beta)\alpha^{n-3}$, then $\alpha^n = (pN_{12}^2 - F_5\beta)\alpha^{n-2}$. Now $K_n = \frac{\alpha^{n-1}-\alpha^{-1}}{a^n}, K_n = \frac{(pN_{12}^2 - F_5\beta)\alpha^{n-3} + (pN_{12}^2 - F_5\beta)\alpha^{n-4}}{a^n}$ and $K_n = \frac{(pN_{12}^2 - F_5\beta)(1+\alpha)}{a-\beta}, n \geq 4$. \[ \square \]

**Theorem 3.2.** If $K_{3c(i,j)} = \binom{2}{2}$ then $F_n = \frac{8(\alpha+\beta)}{a-\beta}, n \geq 3$.

**Proof.** An infinite $K_n$ is related with $F_n$ for every $n$. The explicit formula for $F_n$ is $\frac{a^n-b^n}{a-\beta}$. The Kolakoski string of length $|w| = 3$ over $\Sigma = \{1,2\}$ can be defined as $w_{(i,j)} = \{221\}$ then $K_{(i,j)} = \binom{2}{2}$ or $K_{(i,j)} = \binom{2}{2}$. Then its solution will be $\alpha = -2 - \sqrt{7}$, $\beta = 2 + \sqrt{7}$ and $\alpha = 4\alpha + 3, \beta = 4\beta + 3$. Let $\alpha_{n-1} = 2\alpha^{n-3}$ and $\alpha^n = 4\alpha^{n-3} + 3\alpha^{n-2}$. We know that $F_n = \frac{a^n-b^n}{a-\beta}$. Hence $F_{n-1} + F_{n-2} = \frac{a^n-b^{n-1}+a^{n-2}b^{n-2}}{a-\beta}$ where $F_n = \frac{4(\alpha+\beta)+4(\alpha+\beta)}{a-\beta}, n \geq 3$, and so $F_n = \frac{8(\alpha+\beta)}{a-\beta}, n \geq 3$. \[ \square \]

**Theorem 3.3.** If $\alpha = \frac{5+\sqrt{13}}{2}$, $j > i \geq 2$, then $n > log_\alpha(ij), n \geq 1$.

**Proof.** If the Kolakoski sequence is partitioned as the string of length $|w| = 3$, then $i = 2, 4, \ldots$ and $j = 3, 6, \ldots$, for every $n \geq 1$. Here $n$ is the number of
iterations to compute \((i, j)\). If \(n\) is the total number of iterations then, \(n \neq i + j\), \(n_2 > i + j\), \(n_3 > i + j\), \ldots. If \(w_{(i,j)} = \{112\}\) then \(K_{3c}^{(i,j)} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\). Hence the solution of
\[
\alpha^2 = 5\alpha - 3 \text{ is } \alpha = -1, \beta = \frac{5 + \sqrt{13}}{2}, \gamma = \frac{5 - \sqrt{13}}{2}.
\]
If \(n = 1\), \(\log_\alpha (i) = 0.47, \log_\alpha (j) = 0.75\).
If \(n = 2\), \(\log_\alpha (i) = 0.94, \log_\alpha (j) = 1.22\).
If \(n = 3\), \(\log_\alpha (i) = 1.22, \log_\alpha (j) = 1.5\).
If \(n = 4\), \(\log_\alpha (i) = 1.42, \log_\alpha (j) = 1.7\).
If \(n = 5\), \(\log_\alpha (i) = 1.57, \log_\alpha (j) = 1.85\).
If \(n = 6\), \(\log_\alpha (i) = 1.7, \log_\alpha (j) = 1.98\).
If \(n = 7\), \(\log_\alpha (i) = 1.85, \log_\alpha (j) = 2.08\).
If \(n = 8\), \(\log_\alpha (i) = 1.94, \log_\alpha (j) = 2.17\).
If \(n = 9\), \(\log_\alpha (i) = 2.01, \log_\alpha (j) = 2.25\).
If \(n = 10\), \(\log_\alpha (i) = 2.08, \log_\alpha (j) = 2.33\).

\[
\text{FIGURE 1. For Every } n \log_\alpha (il) \text{ is strictly decreasing}
\]

The Kolakoski string is strictly increasing for every \(n\). Also the ratio of Kolakoski string is strictly decreasing for every \(n\).

\[
4. \text{ PROPOSITION OF } K_n \text{ WITH } F_n
\]

An infinite Kolakoski sequence of symbols \(\Sigma = \{1,2\}\) under \(\sigma_0\) and \(\sigma_1\) are 122112122112211110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101111011110111101

\[ K_{n+1} = \frac{3}{2} K_n - K_{n-1}, \] where \( n = 1, 2, \ldots \) if \( K_0 = 1 \) and \( K_1 = 2 \), \( K_{n+1} = K_{n-1} - K_n \), where \( n = 3, 4, 5, \ldots \) The Kolakoski numbers are related with Fibonacci numbers in many ways, for an example the ratio of Kolakoski strings are approximately equal to 1.618. The decimal fraction of Fibonacci numbers is \( \frac{1}{\sqrt{5}} \) which is equal to 0.11309788686. The decimal fraction of an infinite Kolakoski sequence is \( \frac{1}{\sqrt{5}} \). Further the \((3 \times 3)\) array pattern of recursive relations of Kolakoski sequence is of the form

\[
K = \begin{pmatrix}
K_{n-2} & -F_{n+2} & K_{n+4} \\
K_{n-1} & -F_{n+1} & K_{n+5} \\
K_{n+1} & F_n & -F_{n+1}
\end{pmatrix}, \quad n = 3
\]

Hence \(|K| = 0\) even for the non-palindrome string of Kolakoski array, see \([10, 12]\).

**Theorem 4.1.** If \([X]_n, n \geq 0\) then \([F]_n \approx [K]_n\).

**Proof.** If \( F_n = F_{n-1} + F_{n-2} \), for all \( n \geq 3 \) then the first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, \ldots and the ratio of \( F_n \) is 1.618. An infinite Kolakoski sequence with \( i \) blocks and \( j \) positions over \( \Sigma = \{1,2\} \) under \( \sigma_0 \) and \( \sigma_1 \) are 12211221122111\ldots The Kolakoski string of length \(|w| = 3\) is \( \{122, 112\} \), if \( i = 3n \times (\frac{L}{f_2})^2 \times \phi \) and \( j = 3n \) for ever positive integer.

**Table 1.** Representing \( F_n \) from \( \frac{1}{\sqrt{5}} \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>( F_0 )</td>
<td>( F_1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_n = F_{n-1} + F_{n-2} )</th>
<th>( F_n )</th>
<th>( K(i,j) )</th>
<th>( [K]_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>1</td>
<td>1</td>
<td>(2,3)</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>1</td>
<td>2</td>
<td>(4,6)</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>2</td>
<td>3</td>
<td>(6,9)</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>3</td>
<td>4</td>
<td>(8,12)</td>
</tr>
<tr>
<td>( F_5 )</td>
<td>5</td>
<td>5</td>
<td>(10,15)</td>
</tr>
<tr>
<td>( F_6 )</td>
<td>8</td>
<td>6</td>
<td>(12,18)</td>
</tr>
</tbody>
</table>
The ratio of Kolakoski strings and Fibonacci numbers are approximately equal for every positive integer which is shown in figure 2. The combinations of Kolakoski blocks and positions are made up with the decimal fraction \( \frac{1}{81,891,951,65} \) and the ratio of decimal fraction between \( K_n \) and \( F_n \) is \( 1.086797886 \) is clearly shown in figure 2.

\[\text{Figure 2. Ratio of } [K]_n \text{ and } [F]_n\]

**Theorem 4.2.** For every \( n \geq 6 \), \( K_{3\gamma} \geq \left(\frac{5-\sqrt{13}}{2}\right)^{n-5} \).

**Proof.** If \( K(n) \geq \alpha^{n-5} \), let \( K_{3\gamma} \geq \left(\frac{5-\sqrt{13}}{2}\right)^{n-5} \) where \( \gamma, n \geq 6 \). Let the Kolakoski sequence \( 1, 2, 2, 1, 1, 2, 2, 1, 2, 1, 1, \ldots \)

**Case (i):** If \( n = 6 \), then \( 2 \geq \left(\frac{5-\sqrt{13}}{2}\right)^2 \). So that \( K_n \) is true. Therefore \( K(n + 1) \geq \left(\frac{5-\sqrt{13}}{2}\right)^2 \). Then \( 1 \geq \left(\frac{5-\sqrt{13}}{2}\right)^2 \) is also true.

**Case (ii):** If the proposition is true for all integers, \( \gamma, n \geq 8 \) that is \( K(8), K(9), \ldots \) then \( K_{3(\gamma+1)} \geq \left(\frac{5-\sqrt{13}}{2}\right)^{n-4} \) is true. Let \( \alpha = \frac{5-\sqrt{13}}{2} \), then \( \alpha^2 = 5\alpha - 3 \). So \( \left(\frac{5-\sqrt{13}}{2}\right)^{n-4} = \alpha^{n-4} = 5\alpha^{n-5} - 3\alpha^{n-6} \). And from the recursive relation of Kolakoski sequence, \( K_8 = \frac{3}{2} K_{17} - K_{16} \) and \( 2K_n = 3K_{n-1} - 2K_{n-2}, K_0 = 0 \) and \( K_1 = 2 \). Then \( 3K_{n-1} > 5\alpha^{n-5} \) and \( 6 > 5(\frac{5-\sqrt{13}}{2}) \), \(-2K_{n-2} > -3\alpha^{n-6} \). So that \( K_{3(\gamma+1)} \) is true which implies that \( K_{3\gamma} \geq \left(\frac{5-\sqrt{13}}{2}\right)^{n-5} \). \( \Box \)

**Theorem 4.3.** If \( \phi \) is a Golden ratio, then \( 1 < \frac{\log_{\alpha}(i)}{\log_{\alpha}(j)} < \phi \).

**Proof.** If \( n \) is the number of iterations to compute \( i \) and \( j \) then \( i = 2, 4, \ldots \) and \( j = 3, 6, \ldots \) for every \( n \) if \( |w| = 3 \) over the binary alphabet \( \Sigma = \{1, 2\} \) under \( \sigma_0 \)
and $\sigma_1$ where $\sigma_0$ (Even) = \[
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow 11
\end{align*}
\] and $\sigma_1$ (Odd) = \[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 22.
\end{align*}
\]
An explicit formula for $F_n$ is related with $K_n$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Approaching 1 for every $n$ from $\phi$}
\end{figure}

Therefore, $\log_\phi(i) = 1.440$, $\log_\phi(j) = 2.88$ if $n = 1$, $\log_\phi(i) = 2.88$, $\log_\phi(j) = 3.723$ if $n = 2$. If $n = 3$, $\log_\phi(i) = 3.723$, $\log_\phi(j) = 4.566$. If $n = 4$, $\log_\phi(i) = 4.321$, $\log_\phi(j) = 5.164$. If $n = 5$, $\log_\phi(i) = 4.785$, $\log_\phi(j) = 5.627$. Consequently we can obtain $\frac{\log_\phi(i)}{\log_\phi(j)} < \phi$ that is approaching 1 from $\phi$ approximately for every $n$. \hfill \Box

5. Conclusion

An explicit formula for $K_n$ is derived from $F_n$ for every $n$. The recursive formula for $K_n$ is inductively true for every $n$ is proved. The Kolakoski string is strictly increasing for every $n$ is shown. And the ratio of Kolakoski string $(i, j)$ is approximately equal to $\phi$ is established. Also the decimal fraction of Kolakoski sequence and Fibonacci sequence is described. Future work will be based on finding more relations between Fibonacci and Kolakoski Sequence, it will also focus on associating combinatorial properties of integer sequences.
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