NUMERICAL QUENCHING OF A SEMILINEAR HEAT EQUATION WITH A SINGULAR BOUNDARY OUTFLUX

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Abstract. In this paper, we study the semidiscrete approximation for the following semilinear heat equation with a singular boundary outflux

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u_{xx} + (1 - u)^{-p}, \quad 0 < x < 1, \ t > 0, \\
u_x(0, t) &= 0, \quad u_x(1, t) = -u(1, t)^{-q}, \quad t > 0, \\
u(x, 0) &= u_0(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

We find some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

1. Introduction

In this paper, we consider the semilinear heat equation with a singular boundary outflux

\[
(1.1) \quad u_t = u_{xx} + (1 - u)^{-p}, \quad 0 < x < 1, \ t > 0,
\]

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\[ u_x(0, t) = 0, \quad u_x(1, t) = -u(1, t)^{-q}, \quad t > 0, \]

\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \]

where \( p, q \) are positive constants and \( u_0 : [0, 1] \rightarrow (0, 1) \) is nonincreasing and satisfies the compatibility conditions:

\[ u'_0(0) = 0, \quad u'_0(1) = -u_0(1)^{-q}. \]

**Definition 1.1.** We say that the solution \( u \) of (1.1)–(1.3) quenches in a finite time if there exists a finite time \( T_q \) such that \( \|u(., t)\|_\infty < 1 \) for \( t \in [0, T_q) \) but

\[ \lim_{t \to T_q} \|u(., t)\|_\infty = 1, \]

where \( \|u(., t)\|_\infty = \max\{|u(x, t)| : 0 \leq x \leq 1\} \). The time \( T_q \) is called quenching time of the solution \( u \).

The theoretical study of quenching problems with various boundary has been the subject of investigations of many authors (see [2], [3], [4], [5], [7], [8], [9], [12] and the references cited therein). In [12], B. Selcuk and N. Ozalp prove a finite-time quenching for the solution of (1.1)–(1.3) Under certain conditions, they show that \( x = 0 \) is the only quenching point and they get a quenching rate and a lower bound for the quenching time. In this paper, we are interested in the numerical study using a semidiscrete scheme of (1.1)–(1.3). For previous study on numerical approximations of parabolic system with non-linear boundary conditions we refer to ( [6], [10], [11] )

We organise this paper as follows: In the next section, we give some lemmas which will be used throughout the paper. In the fourth section, under some hypotheses, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fifth section, we give a result about the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

### 2. The Semidiscrete Problem

Let \( I \) be a nonnegative integer, we set \( h = \frac{1}{I} \), and we define the grid, \( x_i = ih, \quad i = 0, \ldots, I \). We approximate the solution \( u \) of the problem (1.1)–(1.3) by the
solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$. For $t \in (0, T_0)$, we have

\begin{equation}
\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + (1 - U_i(t))^{-p}, \quad 0 \leq i \leq I - 1,
\end{equation}

\begin{equation}
\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + (1 - U_I(t))^{-p} - \frac{2}{h} U_I(t)^{-q},
\end{equation}

\begin{equation}
U_i(0) = \varphi_i, \quad 0 \leq i \leq I,
\end{equation}

where

\begin{align*}
\delta^+ \varphi_i &= \frac{\varphi_{i+1} - \varphi_i}{h}, \quad 0 \leq i \leq I - 1, \\
\delta^2 U_i(t) &= \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I - 1, \\
\delta^2 U_0(t) &= \frac{2U_1(t) - 2U_0(t)}{h^2}, \\
\delta^2 U_I(t) &= \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.
\end{align*}

\section{Properties of the Semidiscrete Problem}

In this section, we give some important results which will be used later.

**Lemma 3.1.** Let $b_h(t) \in C^0([0, T], \mathbb{R}^{I+1})$ and $V_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ such that

\begin{equation}
\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0, T],
\end{equation}

\begin{equation}
V_i(0) \geq 0, \quad 0 \leq i \leq I.
\end{equation}

Then we have $V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in [0, T]$.

**Proof.** Let $T_0 < T$. Define the vector $Z_h(t) = e^\lambda V_h(t)$ where $\lambda$ is such that $b_i(t) - \lambda \geq 0 \forall t \in [0, T_0]$.

Let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} Z_i(t)$. $Z_i(t)$ is continuous on the compact $[0, T_0]$, there exists $i_0 \in \{0, \ldots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$.

We observe that:

\begin{equation}
\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,
\end{equation}

\begin{equation}
\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I - 1,
\end{equation}

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We observe that:

\[ \delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0, \]

\[ \delta^2 Z_{i_0}(t_0) = \frac{2Z_{t-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I. \]

Moreover, by a straightforward computation, we get

\[ \frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \]

Using (3.3)–(3.6), we deduce from (3.7) that \((b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0\), which implies that \(Z_{i_0}(t_0) \geq 0\). We deduce that \(V_h(t) \geq 0\), \(\forall t \in [0, T]\) and the proof is complete. \(\square\)

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

**Lemma 3.2.** Let \(g \in C^0(\mathbb{R}, \mathbb{R})\) and \(V_h(t), W_h(t) \in C^1([0, T], \mathbb{R}^{I+1})\), such that for \(0 \leq i \leq I\)

\[ \frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t)) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t)), t \in (0, T) \]

\[ V_i(0) < W_i(0). \]

Then \(V_i(t) < W_i(t), 0 \leq i \leq I, t \in [0, T]\).

**Proof.** Let \(Z_h(t)\) a vector such that \(Z_i(t) = W_i(t) - V_i(t)\) and let \(t_0\), be the first \(t > 0\) such that \(Z_i(t_0) > 0\), \(\forall t \in [0, t_0]\) but \(Z_{i_0}(t_0) = 0\) for a certain \(i_0 \in \{0, \ldots, I\}\).

We observe that:

\[ \frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I, \]

\[ \delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, 1 \leq i_0 \leq I - 1, \]

\[ \delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0, \]

\[ \delta^2 Z_{i_0}(t_0) = \frac{2Z_{t-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I. \]

Which implies that

\[ \frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + g(W_{i_0}(t_0)) - g(V_{i_0}(t_0)) \leq 0. \]
This inequality contradicts (3.8) which ends the proof.

Lemma 3.3. Let $U_h$ be the solution of (2.1)–(2.3). We assume that the initial data at (2.3) satisfies $\varphi_i > 0$, $0 \leq i \leq I$. Then for $t \in (0, T_q^h)$ and $0 \leq i \leq I$, we have

$$U_i(t) > 0.$$  

Proof. Let $t_0$, be the first $t > 0$ such that $U_{i_0}(t) > 0$, $\forall t \in (0, t_0)$ but $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that $i_0$ is the smallest integer checking the inequality above. We observe that

$$0 \leq i_0 \leq I,$$  

and

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} < 0,$$  

for $0 \leq i_0 \leq I - 1,$

$$\frac{\delta^2 U_{i_0}(t_0)}{h^2} = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0,$$  

for $0 \leq i_0 \leq I - 1,$

$$\frac{\delta^2 U_{i_0}(t_0)}{h^2} = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} > 0,$$  

for $i_0 = 0,$

$$\frac{\delta^2 U_{i_0}(t_0)}{h^2} = \frac{2U_{I-2}(t_0) - 2U_{I-1}(t_0)}{h^2} > 0,$$  

for $i_0 = I.$

By a straightforward computation, we get

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) - (1 - U_{i_0}(t_0))^{-p} < 0,$$  

for $0 \leq i_0 \leq I - 1,$

$$\frac{dU_1(t)}{dt} - \delta^2 U_1(t) - (1 - U_1(t))^{-p} + \frac{2}{h}U_1(t)^{-q} < 0.$$  

But these inequalities contradict (2.1)–(2.2) and this proof is complete. 

Lemma 3.4. Let $U_h$ be the solution of (2.1)–(2.3). Then we have for $t \in [0, T_q^h)$

$$U_i(t) > U_{i+1}(t), 0 \leq i \leq I - 1.$$  

Proof. Introduce the vector $Z_h(t)$ such that $Z_i(t) = U_i(t) - U_{i+1}(t)$ for $t \in [0, T_q^h)$, $i = 0, \ldots, I - 1.$ Let $t_0$, be the first $t > 0$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that $i_0$ is the smallest integer checking the inequality above. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$  

for $0 \leq i_0 \leq I - 1,$

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Lemma 3.5. Let $U_h$ be the solution of (2.1)–(2.3). Then we have

$$\frac{dZ_i(t)}{dt} > 0, \ 0 < i \leq I, \ t \in (0, T^h_q).$$

Proof. Consider the vector $Z_h(t)$ such that $Z_i(t) = \frac{dU_i(t)}{dt}, \ t \in (0, T^h_q), \ i = 0, \ldots, I$. Let $t_0$, be the first $t \in (0, T^h_q)$ such that $Z_i(t) > 0, \ \forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that $i_0$ is the smallest integer checking the inequality above. We observe that:

\begin{align}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,
\end{align}

\begin{align}
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 0 < i_0 < I - 1,
\end{align}

\begin{align}
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,
\end{align}

\begin{align}
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{I-1}(t_0) - 2Z_1(t_0)}{h^2} > 0, \quad i_0 = I.
\end{align}

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - p(1 - U_{i_0}(t_0))^{-p-1}Z_{i_0}(t_0) < 0, \quad 0 \leq i_0 \leq I - 1,$$

where $\zeta_{i_0}$ is an intermediate value between $U_i(t)$ and $U_{i+1}$. And

$$\frac{dZ_{i_0+1}(t_0)}{dt} - \delta^2 Z_{i_0+1}(t_0) - \frac{2}{h} U_i(t)^{-q} - p(1 - \theta_i(t_0))^{-p-1}Z_{i_0+1}(t_0) < 0.$$

Where $\theta_i$ is an intermediate value between $U_{i-1}(t)$ and $U_i$. But these inequalities contradict (2.1)–(2.2) and this proof is complete. \hfill \Box
\[
\frac{dZ_I(t_0)}{dt} - \delta^2 Z_I(t_0) - p(1 - U_I(t_0))^{-p-1}Z_I(t_0) - \frac{2}{\delta} q U_I(t)^{-q-1}Z_I(t_0) < 0.
\]

But these inequalities contradict (2.1)–(2.2) and this proof is complete. \qed

4. Quenching Solutions

In this section, we show that under some assumptions, the solution \(U_h\) of (2.1)–(2.3) quenches in a finite time and estimate its semidiscrete quenching time.

**Lemma 4.1.** Let \(U_h \in \mathbb{R}^{I+1}\) such that \(\|U_h\|_\infty < 1\) and let \(p\) be a positive constant. Then, we have

\[
\delta^2 (1 - U_i)^{-p} \geq p(1 - U_i)^{-p-1}\delta^2 U_i, \quad 0 \leq i \leq I.
\]

**Proof.** Let us introduce \(f(s) = (1 - s)^{-p}\). We observe that \(f\) is a convex function for nonnegative values of \(s\). Apply Taylor’s expansion to obtain

\[
\begin{align*}
\delta^2 f(U_0) &= f'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} f''(\theta_0), \\
\delta^2 f(U_i) &= f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} f''(\theta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} f''(\eta_i), \quad 1 \leq i \leq I - 1, \\
\delta^2 f(U_I) &= f'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} f''(\eta_I),
\end{align*}
\]

where \(\theta_i\) is an intermediate between \(U_i\) and \(U_{i+1}\) and \(\eta_i\) the one between \(U_{i-1}\) and \(U_i\). Use the fact that \(\|U_h\|_\infty < 1\) to complete the proof. \qed

**Theorem 4.1.** Let \(U_h\) be the solution of (2.1)–(2.3), and assume that there exist a nonnegative constant \(A\) such that the initial data at (2.3) satisfies

\[
\begin{align*}
\delta^2 \varphi_i + (1 - \varphi_i)^{-p} &\geq A(1 - \varphi_i)^{-p}, \quad 0 \leq i \leq I - 1, \\
\delta^2 \varphi_I + (1 - \varphi_I)^{-p} - \frac{2}{\delta} q \varphi_I^{-q} &\geq A(1 - \varphi_I)^{-p}.
\end{align*}
\]

Then, the solution \(U_h\) quenches in a finite time \(T'^h_q\) and we have the following estimate

\[
T'^h_q \leq \frac{(1 - \|\varphi_h\|_\infty)^{p+1}}{A(p + 1)}.
\]
Lemma 3.1 that

By a straightforward computation we get

\[ J_i(t) = \frac{dU_i(t)}{dt} - A(1 - U_i(t))^{-p}, \quad 0 \leq i \leq I. \]  

(4.3)

Consider the function

\[ J \]

which implies that

From Lemma 4.1, we have

\[ A\delta^2(1 - U_i(t))^{-p} \geq pA(1 - U_i(t))^{-p-1}\delta^2U_i(t), \quad 0 \leq i \leq I. \]

Which implies that

\[
\frac{dJ_i(t)}{dt} - \delta^2J_i(t) \geq p(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt} - pA(1 - U_i(t))^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2U_i(t)), \\
0 \leq i \leq I - 1,
\]

\[
\frac{dJ_i(t)}{dt} - \delta^2J_i(t) \geq p(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt} + \frac{2q}{h}U_i^q(t)\frac{dU_i(t)}{dt} \\
- pA(1 - U_i(t))^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2U_i(t)).
\]

We deduce that

\[
\frac{dJ_i(t)}{dt} - \delta^2J_i(t) \geq p(1 - U_i(t))^{-p-1}J_i(t), \quad 0 \leq i \leq I - 1,
\]

\[
\frac{dJ_i(t)}{dt} - \delta^2J_i(t) \geq p(1 - U_i(t))^{-p-1}J_i(t) + \frac{2q}{h}U_i(t)^q(t)\frac{dU_i(t)}{dt} + pA(1 - U_i(t))^{-p-1}U_i(t)) + pA(1 - U_i(t))^{-p-1}U_i(t).)
\]

From (4.1)–(4.2), we observe that \( J_i(0) \geq 0 \) for \( 0 \leq i \leq I \). We deduce from Lemma 3.1 that \( J_i(t) \geq 0, 0 \leq i \leq I \), which implies that

\[ dU_i(t) \geq A(1 - U_i(t))^{-p}dt, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h). \]

Integrating the above inequalities over the interval \([t, T_q^h)\), we get

\[ T_q^h - t \leq \frac{(1 - U_i(t))^{p+1}}{A(p + 1)}, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h). \]  

(4.4)

Taking \( t = 0 \), we obtain:

\[ T_q^h \leq \frac{(1 - \varphi_i)^{p+1}}{A(p + 1)}, \quad 0 \leq i \leq I. \]
Using the fact that \( \| \varphi_h \|_\infty = \varphi_0 \), we get:
\[
T_q^h \leq \frac{(1 - \| \varphi_h \|_\infty)^{p+1}}{A(p + 1)}.
\]
We have the desired result. \( \square \)

**Remark 4.1.** Integrating the inequality (4.4) over interval \([t_0, T_q^h]\), we have
\[
T_q^h - t_0 \leq \frac{(1 - U_i(t_0))^{p+1}}{A(p + 1)}, \quad t_0 \in [0, T_q^h), \quad 0 \leq i \leq I
\]
and
\[
\| U_h \|_\infty \leq 1 - C_1(T_q^h - t_0)^{\frac{1}{p+1}},
\]
where \( C_1 = (A(p + 1))^{\frac{1}{p+1}} \).

The Remark 4.1 is crucial to prove the convergence of the semidiscrete quenching time.

## 5. Convergence of Semidiscrete Quenching Times

**Theorem 5.1.** Assume that the problem (1.1)–(1.3) has a solution \( u \in C^{1,1}([0, 1] \times [0, T]) \) such that \( \sup_{t \in [0, T]} \| u \| = \lambda < 1 \) and the initial data at (2.3) verifies
\[
\| \varphi_h - u_h(0) \|_\infty = o(1) \quad \text{as} \quad h \to 0,
\]
where \( u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T, \ t \in [0, T] \). Then, for \( h \) small enough, the semidiscrete problem (2.1)–(2.3) has a unique solution \( U_h \in C^1([0, T], \mathbb{R}^{I+1}) \) such that
\[
\max_{t \in [0,T]} (\| U_h(t) - u_h(t) \|_\infty) = O(\| \varphi_h - u_h(0) \|_\infty + h) \quad \text{as} \quad h \to 0.
\]

**Proof.** Let \( \rho > 0 \) be such that \( \rho + \lambda < 1 \). The problem (2.1)–(2.3) has for each \( h \), a unique solution \( U_h \in C^1([0, T], \mathbb{R}^{I+1}) \). Let \( t(h) \) the greatest value of \( t > 0 \) such that
\[
\| U_h(t) - u_h(t) \|_\infty < \rho \quad \text{for} \quad t \in (0, t(h)).
\]
The relation (5.1) implies that \( t(h) > 0 \) for \( h \) small enough. Let \( t^*(h) = \min\{t(h), T\} \). By the triangular inequality, we obtain
\[
\| U_h(t) \|_\infty \leq \| u(\cdot, t) \|_\infty + \| U_h(t) - u_h(t) \|_\infty \quad \text{for} \quad t \in (0, t^*(h)),
\]
which implies that
\[
\|U_h(t)\|_{\infty} \leq \lambda + \rho, \quad \text{for } t \in (0, t^*(h)).
\]

Let \( e_h(t) = U_h(t) - u_h(t) \) be the error of discretization. Using Taylor’s expansion, we have for \( t \in (0, t^*(h)) \),
\[
\frac{d e_0(t)}{dt} - \delta^2 e_0(t) = p(1 - \beta_0)^{-p-1} e_0(t) + h \left( \frac{h}{12} u_{xxxx}(\tilde{x}_0, t) + \frac{2}{3} u_{xxx}(x_0, t) \right),
\]
\[
\frac{d e_i(t)}{dt} - \delta^2 e_i(t) = p(1 - \beta_i(t))^{-p-1} e_i(t) + \frac{h^2}{12} u_{xxxx}(\tilde{x}_i, t), \quad 1 \leq i \leq I - 1
\]
\[
\frac{d e_I(t)}{dt} - \delta^2 e_I(t) = \left( p(1 - \beta_I)^{-p-1} + \frac{2q}{h} \mu_I^{-q-1}(t) \right) e_I(t)
\]
\[
+ h \left( \frac{h}{12} u_{xxxx}(\tilde{x}_I, t) - \frac{2}{3} u_{xxx}(x_I, t) \right),
\]

where \( \beta_i(t) \) is an intermediate value between \( U_i(t) \) and \( u_i(t) \), \( 0 \leq i \leq I \) and \( \mu_I(t) \) the one between \( U_I(t) \) and \( u_I(t) \).

Using (5.3), there exist nonnegative constants \( K, M \) such that
\[
\frac{d e_0(t)}{dt} - \delta^2 e_0(t) \leq M|e_0(t)| + Kh.
\]

(5.4)
\[
\frac{d e_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Kh^2, \quad 1 \leq i \leq I - 1,
\]

(5.5)
\[
\frac{d e_I(t)}{dt} - \delta^2 e_I(t) \leq \frac{M}{h} |e_I(t)| + Kh.
\]

(5.6)

Let \( Z_h(t) \) the vector defined by
\[
Z_i(t) = e^{(M+1)t}(||\varphi_h - u_h(0)||_{\infty} + Kh), \quad 0 \leq i \leq I.
\]

A simple calculation give
\[
\frac{d Z_0(t)}{dt} - \delta^2 Z_0(t) > M|Z_0(t)| + Kh,
\]

(5.7)
\[
\frac{d Z_i(t)}{dt} - \delta^2 Z_i(t) > M|Z_i(t)| + Kh^2, \quad 1 \leq i \leq I - 1,
\]

(5.8)
\[
\frac{d Z_I(t)}{dt} - \delta^2 Z_I(t) > \frac{M}{h} |Z_I(t)| + Kh,
\]

(5.9)
\[
Z_i(0) > e_i(0), \quad 0 \leq i \leq I.
\]

(5.10)
From Lemma 3.2, we obtain
\[ Z_i(t) > e_i(t), \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I. \]

By analogy, we also prove that
\[ Z_i(t) > -e_i(t), \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I. \]

Hence we have
\[ Z_i(t) > |e_i(t)|, \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I. \]

We deduce that
\[ \|U_h(t) - u_h(t)\|_{\infty} \leq (\|\varphi_h - u_h(0)\|_{\infty} + Kh)e^{(M+1)t}, \quad t \in (0, t^*(h)). \]

Next we prove that \( t^*(h) = T \). Suppose that \( t(h) < T \) From (5.2), we obtain
\[ \rho \leq 2 0 2 \]
\[ \text{Since } (\|\varphi_h - u_h(0)\|_{\infty} + Kh)e^{(M+1)T} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad \text{we deduce from (5.11) that } \rho \leq 0 \text{ which is impossible and we conclude the proof.} \]

**Theorem 5.2.** Suppose that the solution \( u \) of problem (1.1)-(1.3) quenches in a finite time \( T_q \) such that \( u \in C^{4,1}([0, 1] \times [0, T_q]) \) and the initial data at (2.3) satisfies
\[ \|\varphi_h - u_h(0)\|_{\infty} = o(1) \quad \text{as } h \rightarrow 0. \]

Under the assumptions of Theorem 4.1, the solution \( U_h \) of problem (2.1)-(2.3) quenches in finite time \( T_q^h \) and we have
\[ \lim_{h \rightarrow 0} T_q^h = T_q. \]

**Proof.** Set \( \varepsilon > 0 \). There exists \( \eta > 0 \) such that
\[ \frac{(1 - \varrho)^{p+1}}{A(p + 1)} < \frac{\varepsilon}{2}, \quad 0 \leq \varrho \leq \eta. \]

Since \( u \) quenches in a finite time \( T_q \), there exists a time \( T_0 < T_q \) such that \( |T_0 - T_q| < \frac{\varepsilon}{2} \) and \( 0 \leq \|u(\cdot, t)\|_{\infty} \leq \frac{\eta}{2} \) for \( t \in [T_0, T_q] \). Setting \( T_1 = \frac{T_0 + T_q}{2} \), it is not hard to see that \( \|u(\cdot, t)\|_{\infty} < 1 \) for \( t \in [0, T_1] \).

From Theorem 5.1, we have \( \|U_h(T_1) - u_h(T_1)\|_{\infty} \leq \frac{\eta}{2} \). Applying the triangle inequality, we get
\[ \|U_h(T_1)\|_{\infty} \leq \|U_h(T_1) - u_h(T_1)\|_{\infty} + \|u_h(T_1)\|_{\infty} \leq \eta. \]
From Theorem 4.1, \( U_h \) quenches in a finite time \( T_h \). We deduce from Remark 4.1 and (5.12) that
\[
|T_h - T_q| \leq |T_h - T_1| + |T_1 - T_q| \leq \frac{(1 - U_h(T_1))^{p+1}}{A(p + 1)} + \frac{\epsilon}{2} \leq \epsilon.
\]
\[
\Box
\]

6. Numerical experiments

In this section, we present some numerical approximations of the quenching time of the problem (1.1)–(1.3) in the case where \( u_0(x) = 0.7 - \frac{1}{2}x^4, p = 8.03 \) and \( q = -\log(2)/\log(0.2) \). Firstly, we consider the following explicit scheme
\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_i^{(n+1)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (1 - U_i^{(n)})^{-p}, \quad 1 \leq i \leq I - 1,
\]
\[
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - U_0^{(n)})^{-p},
\]
\[
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (1 - U_I^{(n)})^{-p} - \frac{2}{h}(U_I^{(n)})^{-q},
\]
\[
U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,
\]
where \( n \geq 0, \Delta t_n = \frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_{\infty})^{p+1} \). We also consider the implicit scheme
\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_i^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + (1 - U_i^{(n)})^{-p}, \quad 1 \leq i \leq I - 1,
\]
\[
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_{1}^{(n+1)} - 2U_{0}^{(n+1)}}{h^2} + (1 - U_0^{(n)})^{-p},
\]
\[
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_{I}^{(n+1)}}{h^2} + (1 - U_I^{(n)})^{-p} - \frac{2}{h}(U_I^{(n)})^{-q},
\]
\[
U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,
\]
where \( n \geq 0, \Delta t_n = h^2(1 - \|U_h^{(n)}\|_{\infty})^{p+1} \). In the following tables, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes 16, 32, 64, 128, 256, 512. The
numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when
\[ |T^{n+1} - T^n| \leq 10^{-16}. \]
The order $s$ of the method is computed from
\[ s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}. \]
For the discret initial data we take $\varphi_i = 0.7 - \frac{1}{2}(i h)^2$.

**Table 1.** Numerical quenching times obtained with the explicit Euler method $p = 8.03$ and $q = -\log(2)/\log(0.2)$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T^n$</th>
<th>$n$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.000002136</td>
<td>578</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.000002111</td>
<td>2158</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.000002104</td>
<td>8006</td>
<td>2.01</td>
</tr>
<tr>
<td>128</td>
<td>0.000002103</td>
<td>29512</td>
<td>2.00</td>
</tr>
<tr>
<td>256</td>
<td>0.000002103</td>
<td>107986</td>
<td>2.00</td>
</tr>
<tr>
<td>512</td>
<td>0.000002102</td>
<td>391701</td>
<td>1.97</td>
</tr>
</tbody>
</table>

**Table 2.** Numerical quenching times obtained with the implicit Euler method $p = 8.03$ and $q = -\log(2)/\log(0.2)$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T^n$</th>
<th>$n$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.000002136</td>
<td>578</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.000002111</td>
<td>2158</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.000002104</td>
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<td>128</td>
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<td>107986</td>
<td>1.99</td>
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<tr>
<td>512</td>
<td>0.000002102</td>
<td>391701</td>
<td>1.88</td>
</tr>
</tbody>
</table>

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$, $p = 8.03$ and $q = -\log(2)/\log(0.2)$. In figures 1, 2 and figures 3, 4,
we can appreciate that the discrete solution is nonincreasing and reaches the value one at the first node. In figures 5 and 6, we see that the approximation of $\|U_h^{(n)}\|_\infty$ is nondecreasing and tends to the value one when $t$ tends to $2.5 \times 10^{-6}$.

**Figure 1.** Evolution of the numerical solution (explicit scheme).

**Figure 2.** Evolution of the numerical solution (implicit scheme).

**Figure 3.** The profile of the approximation of $u(x,T)$ where, $T$ is the quenching time (explicit scheme).

**Figure 4.** The profile of the approximation of $u(x,T)$ where, $T$ is the quenching time (implicit scheme).
Figure 5. The profile of the approximation of $\| U_h^{(n)} \|_\infty$ (explicit scheme).

Figure 6. The profile of the approximation of $\| U_h^{(n)} \|_\infty$ (implicit scheme).

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