ON THE LIMITS OF SOME $p$-ADIC SCHNEIDER CONTINUED FRACTIONS

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ABSTRACT. In the present paper, we first generalize some convergence results for continued fractions given in real domain and $p$-adic domain. However, we prove the transcendence of a $p$-adic number given by it's Schneider continued fractions, such that the sequence of partial quotients is a Thue-Morse sequence.

1. INTRODUCTION AND STATEMENTS

Schneider continued fractions are defined as sequences of the shape:

$$(1.1)\quad \frac{p_n}{q_n} = a_0 + \frac{p^{\alpha_0}}{a_1 + \frac{p^{\alpha_1}}{a_2 + \cdots \frac{p^{\alpha_{n-1}}}{a_n}}}$$

where $(\alpha_i)_{i\in\mathbb{N}}$ is a sequence of positive integers and $(a_i)_{i\in\mathbb{N}}$ is a sequence of integers in $\{1, \ldots, p-1\}$.

In [7], M. Kojima proved the convergence of (1.1), both in $\mathbb{Q}_p$ and in $\mathbb{R}$ when $\alpha_i = 1$ for all $i \in \mathbb{N}$ (he proved indeed slightly more, considering that the coefficients $(a_i)_{i\in\mathbb{N}}$ could be any positive integers).

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Moreover, he proved the convergence in $\mathbb{Q}_p$ for all sequences of $(\alpha_i)_{i \in \mathbb{N}}$. However, he failed to prove a similar theorem for the convergence in $\mathbb{R}$.

In our paper, we first generalize its convergence result in $\mathbb{R}$, easily to bounded sequences of $(\alpha_i)_{i \in \mathbb{N}}$ (see theorem 2.1) and with much more tedious method to more general sequences (see theorem 2.2 up to theorem 2.5). Furthermore, we prove transcendence results concerning the limits in $\mathbb{Q}_p$ and in $\mathbb{R}$, when the sequence $(\alpha_i)_{i \in \mathbb{N}}$ is a Thue-Morse sequence (see Main Theorem 1.1). The method is based on Schlickewei theorem [8] providing a sufficient condition of transcendence.

To state our results, we will recall some definitions and basic facts from $p$-adic numbers and words. Throughout, $p$ is a prime number, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{Q}^*$ is the field of nonzero rational numbers and $\mathbb{R}$ is the field of real numbers. We use $|.|$ to denote the ordinary absolute value, $v_p$ the $p$-adic valuation, $|.|_p$ the $p$-adic absolute value. The field of $p$-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value.

Let us introduce the combinatorics to be used in the sequel: Let the word $W = w_1 w_2 \ldots w_{m-1} w_m$ be on the alphabet $A$, we denote $|W|$ the length $m$ of $W$. The mirror of $W$ is the word $\overline{W} = w_m w_{m-1} \ldots w_2 w_1$. We say that $W$ is a palindrome if $W = \overline{W}$.

A Thue-Morse sequence $(t_n)_{n \in \mathbb{N}}$ with values in a two elements set $\{\alpha; \beta\}$ is defined by $t_n = \alpha$ (resp. $\beta$) if the binary expansion of $n$ has an even (resp. odd) number of digits 1. We shall identify a sequence $(a_k)_{k \in \mathbb{N}}$ of elements of a given set $A$ with an infinite word $a_0 a_1 \ldots a_k \ldots$ in $A^*$. A Thue-Morse sequence has numerous properties (see [2]). In the sequel, the following are used:

**Theorem 1.1.** Let $(t_n)_{n \in \mathbb{N}}$ be a Thue-Morse sequence. Then the word $t_0 t_1 \ldots t_{4^{k-1}}$ is a palindrome and the two letters of the alphabet have the same number of occurrences.

The transcendence method of Adamczewski and Bugeaud [1] is based on the Schmidt’s subspace theorem. We make use the $p$-adic version of this theorem (see [8]). Let $\nu \geq 2$ be an integer, $x = (x_1, \ldots, x_\nu)$ a $\nu$-tuple of rational numbers. We put $|x|_\infty = \max\{|x_i|; 1 \leq i \leq \nu\}$ and $|x|_p = \max\{|x_i|_p; 1 \leq i \leq \nu\}$.

**Theorem 1.2** (Schlickewei). Let $p$ be a prime number, $L_1, \ldots, L_{\nu, \infty}$ be $\nu$ linearly independent forms with variable $x$ and algebraic real coefficients, $L_1, p, \ldots, L_{\nu, p}$ be $\nu$ linearly independent forms with algebraic $p$-adic coefficients and same variables
and \( \delta > 0 \) a real number. Then, the set of solutions \( x \) in \( \mathbb{Z}^\nu / \{0\} \) of the inequality
\[
\prod_{i=1}^\nu \left( |L_{i,\infty}(x)| \cdot |L_{i,p}(x)|_p \right) \leq |x|^{-\delta}
\]
is contained in the union of a finite number of proper subspaces of \( \mathbb{Q}^\nu \).

In [3], we have studied the periodicity of rational number given by its \( p \)-adic expansion. So, in [4], we have studied the transcendence of a \( p \)-adic number given by its Ruban continued fractions, such that the sequence of partial quotients is of Thue-Morse.

**Theorem 1.3.** Let \( p \) be a prime odd positive integer. Let \( \alpha = \frac{\alpha_1}{\alpha_2} \) and \( \beta = \frac{\beta_1}{\beta_2} \) be two rational numbers in \( \mathbb{Z} \left[ \frac{1}{p} \right] \cap (0; p) \) such that \( v_p(\alpha_1) = v_p(\beta_1) = 0 \) and \( v_p(\alpha_2) \geq v_p(\beta_2) \geq 1 \). Let \( \theta \) be defined in \( \mathbb{Q}_p \) as the limit of \( [0; a_1, a_2, \ldots] \) where \( a_i \in \{\alpha, \beta\} \). Suppose that the sequence of partial quotients \( (a_i)_{i \geq 1} \) is a Thue-Morse word. Let us denote \( \Xi = \max \{\alpha; \beta\} \). If
\[
p^{\min\{v_p(\beta_2) - v_p(\alpha_2)\}} > \max\{\alpha_2; \beta_2\} \times \Xi + \frac{\Xi + \sqrt{\Xi^2 + 4}}{2},
\]
then, the \( p \)-adic number \( \theta \) is either transcendental or quadratic.

Using the \( p \)-adic version of the subspace theorem, we give sufficient conditions for a number defined through a Schneider continued fraction to be quadratic or transcendental.

**Main Theorem 1.1.** Let \( p \) be a prime odd positive integer. Let \( ((a_i;\alpha_i))_{i \in \mathbb{N}} \) with values in \( \{1, \ldots, p-1\} \times \mathbb{N}^\ast \). We suppose the sequence \( ((\alpha_i))_{i \in \mathbb{N}} \) is bounded, and let \( A = \max\{\alpha_i; i \in \mathbb{N}\} \). Let \( \theta \) be defined in \( \mathbb{Q}_p \) as the limit of
\[
\theta = a_0 + \frac{p^{a_0}}{a_1} + \frac{p^{a_1}}{a_2} + \cdots + \frac{p^{a_{n-1}}}{a_n} + \cdots
\]
Suppose that the sequence of partial quotients \( (a_i)_{i \geq 1} \) is a Thue-Morse word. then \( \theta \) is either transcendental or quadratic.
2. CONTINUED FRACTIONS

Definitions and results of this section are well known (see [6] for the real case and [5,9] for the $p$-adic case), so we just sketch the proofs.

Definition 2.1. From a sequence $((a_i, \alpha_i))_{i \in \mathbb{N}}$ with values in $\{1, \ldots, p-1\} \times \mathbb{N}^*$, we define a sequence of homographic functions of a field $K = \mathbb{R}$ or $\mathbb{Q}_p$ by

$$[([a, \alpha]; x] = \frac{a}{x} + p^\alpha x$$

and

$$[([a_0, \alpha_0], [a_1, \alpha_1], \ldots, [a_n, \alpha_n]); x] = [([a_0, \alpha_0], [a_1, \alpha_1], \ldots, [a_n, \alpha_n]; [a_n, \alpha_n]; x]].$$

We call $[([a_0, \alpha_0], [a_1, \alpha_1], \ldots, [a_{n-1}, \alpha_{n-1}], [a_n]); x]$ the $n$-th convergent of this sequence.

A matrix of the homographic function $\frac{a_k + p^\alpha_k x}{x}$ is

$$\begin{pmatrix} a_k & p^\alpha_k \\ 1 & 0 \end{pmatrix}.$$

Hence, a matrix of the homographic function $[([a_0, \alpha_0], [a_1, \alpha_1], \ldots, [a_n, \alpha_n]); x]$ is

$$\begin{pmatrix} a_0 & p^\alpha_0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_1 & p^\alpha_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & p^\alpha_n \\ 1 & 0 \end{pmatrix}.$$

Let us denote it $\begin{pmatrix} p_n & p'_n \\ q_n & q'_n \end{pmatrix}$. We have

$$a_0 + \frac{p^\alpha_0}{a_1 + \frac{p^\alpha_1}{a_2 + \cdots + \frac{p^\alpha_n}{a_n + \frac{p^\alpha_n}{x}}}} = [(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_n, \alpha_n); x] = \frac{p_n x + p'_n}{q_n x + q'_n}.$$

The sequences $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ satisfy both the following recurrence relation:

$$u_{n+2} = a_{n+2} u_{n+1} + u_n p^\alpha_{n+1},$$

with $p_{-1} = 1$, $p_0 = a_0$ and $q_{-1} = 0$, $q_0 = 1$. Moreover, we have $p'_{n+1} = p_n p^\alpha_{n+1}$ and $q'_{n+1} = q_n p^\alpha_{n+1}$. Hence we have:

$$a_0 + \frac{p^\alpha_0}{a_1 + \frac{p^\alpha_1}{a_2 + \cdots + \frac{p^\alpha_n}{a_n}}} = \frac{p_n - p'_{n-1}}{p_{n-1} a_n + q'_{n-1}} = \frac{p_n}{q_n}.$$
Given a $p$-adic number $\alpha$, a question is to find a sequence $((a_i, \alpha_i))_{i \in \mathbb{N}}$ with values in $\{1, \ldots, p - 1\} \times \mathbb{N}^*$ such that the sequence $[((a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{n-1}, \alpha_{n-1}); a_n)]_{n \in \mathbb{N}}$ converges to $\alpha$ in $\mathbb{Q}_p$, with a unique solution. We shall concurrently consider the convergence of this sequence in $\mathbb{R}$.

Let us consider the series $\sum_{i=1}^{n} (\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}})$. Using the easy to check property $p_i q_{i-1} - q_i p_{i-1} = (-1)^i \prod_{k=0}^{i-1} p^\alpha_k$, it comes

$$\frac{p_n}{q_n} = \frac{p_0}{q_0} + \sum_{i=1}^{n} (-1)^i \prod_{k=0}^{i-1} p^\alpha_k.$$

**Lemma 2.1.** For all positive $n$, we have: $q_n \geq p_n^2$, et $p_n \geq p_n^2$.

**Proof.** Easy recursion on $n$. □

### 2.1. Convergence in $\mathbb{R}$.

**Theorem 2.1.** If the set $\{\alpha_i; i \in \mathbb{N}\}$ is bounded, the sequence defined by

$$[((a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i)]$$

converges in $\mathbb{R}$.

**Proof.** We shall use the following notation

$$u_i = \prod_{k=0}^{i-1} \frac{p^\alpha_k}{q_k q_{k-1}}.$$

Let us use Leibniz criterium for alternating series. We have

$$\frac{u_i}{u_{i+1}} - 1 = \frac{a_{i+1} q_i}{p^\alpha_i q_i} > 0.$$

Suppose that $A = \max\{\alpha_i; i \in \mathbb{N}\}$. Then $\frac{u_i}{u_{i+1}} - 1 \geq \frac{1}{p^A} > 0$ and the proof is complete. □

In the sequel, we need the following lemma, this subsection we provide some insight convergence in other uses:

**Lemma 2.2.** Let the sequence $(v_n)_{n \in \mathbb{N}}$ be defined by $v_{n+1} = k a^n v_n + v_{n-1}$ with $0 < a < 1$ and $0 < k \leq 1$. If $a < \frac{2}{k + \sqrt{k^2 + 4}}$, then the sequences $v_{2n}$ and $v_{2n+1}$ converge, and their limits are different.
Proof. To show the convergence, let the sequence \((F_n)_{n \in \mathbb{N}}\) defined by

\[ F_{n+1} = kF_n + F_{n-1} \]

with \(F_0 = v_0\) et \(F_1 = v_1\). Then, \(F_n\) is given by the formula

\[ F_n = \frac{1}{2} (v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \Phi^k + \frac{1}{2} (v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \left(\frac{-1}{\Phi}\right)^n, \]

with \(\Phi_k = \frac{k + \sqrt{k^2 + 4}}{2}\). It is easy to show by induction that \(v_n \leq F_n\). In the other hand, we have \(v_{n-1} \leq k\alpha^n F_n + v_{n-1}\) and

\[ k\alpha^n F_n \sim \frac{k}{2} \left(\frac{u_0 + 2u_1 - ku_0}{\sqrt{k^2 + 4}}\right) (a\Phi)^n. \]

The proof is complete where \(\Phi_k < \frac{1}{a}\).

To show that the two limits are different, suppose that \(v_{2n}\) converges to \(\ell\) and \(v_{2n+1}\) converges to \(\ell'\), we have

\[ F_{2n} = \frac{1}{2} (v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \Phi^{2n} + \frac{1}{2} (v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \left(\frac{1}{\Phi}\right)^{2n} \]

and

\[ F_{2n+1} = \frac{1}{2} (v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \Phi^{2n+1} - \frac{1}{2} (v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}) \left(\frac{1}{\Phi}\right)^{2n+1}. \]

Then, for \(\lambda = \frac{2v_1 - kv_0}{\sqrt{k^2 + 4}}\) we have

\[ v_{2n} < \frac{1}{2} (v_0 + \lambda) \Phi^{2n} + \frac{1}{2} (v_0 - \lambda) \left(\frac{1}{\Phi}\right)^{2n} \]

and

\[ v_{2n+1} < \frac{1}{2} (v_0 + \lambda) \Phi^{2n+1} - \frac{1}{2} (v_0 - \lambda) \left(\frac{1}{\Phi}\right)^{2n+1}. \]

It follows that

\[ \frac{v_{2n}}{v_{2n+1}} < \left(\frac{v_0 + \lambda}{v_0 + \lambda}\right) \Phi^{4n} + (v_0 - \lambda) \Phi^{4n+2} + (v_0 - \lambda). \]

Passing to the limit, we obtain \(\ell < \frac{1}{\Phi} \ell' < \ell'\).

Example 1. Suppose that for all \(i\), \(a_i = 1\) and \(\alpha_i = i\). Then the sequence defined by \([\{(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i\}]\) does not converge in \(\mathbb{R}\).

Proof. We put \(q_i' = \frac{a_i}{p^2}\), therefore \(q_{i+1}' = k\alpha' q_i + q_{i-1}'\), with \(a = \frac{1}{p^2}\) and \(k = \frac{1}{p^4}\), so with the previous lemma the sequence \((q_{2i}')_{i}\) converge to the limit \(\ell\) and the
sequence \( (q'_{2i-1})_i \) converge to the limits \( \ell' \), that is to say that for all given \( \epsilon > 0 \) from a certain rank we have
\[
q_{2i} \leq (\ell + \epsilon)p\frac{(2i)^2}{4} \quad \text{and} \quad q_{2i-1} \leq (\ell' + \epsilon)p\frac{(2i-1)^2}{4}
\]
Then
\[
u_{2i} = \frac{p^{\frac{2i(2i-1)}{4}}}{q_{2i}q_{2i-1}} \geq \frac{p^{\frac{1}{2}}}{(\ell + \epsilon)(\ell' + \epsilon)}
\]
thus \( u_{2i} \) do not converge to 0.

In the sequel, we shall use this notation:
\[
A_{2n} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1}
\]
\[
A_{2n+1} = \alpha_0 + \alpha_2 + \cdots + \alpha_{2n}
\]
\[
q'_i = \frac{q_i}{p^A_i}.
\]
Hence we have obviously \( u_i = \frac{1}{q'_i} \), \( A_{i+2} - A_i = \alpha_{i+1} \), \( q'_0 = 1, q'_1 = a_1 \), and
\[
q'_{i+2} = a_{i+2} \frac{q'_{i+1}}{p^{A_{i+2} - A_{i+1}}} + q'_i.
\]

**Theorem 2.2.** Suppose that for all \( i, a_i \in \{1, \ldots, p-1\} \) and \( \alpha_i = i \). Then the sequence \( [(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i] \) does not converges in \( \mathbb{R} \).

**Proof.** We have \( A_{2n} = 1 + 3 + \cdots + (2n - 1) = n^2 \), \( A_{2n+1} = 2 + 4 + \cdots + (2n) = n(n + 1) \), \( A_{i+2} - A_i = i + 1 \) and we have \( A_{i+2} - A_{i+1} = \frac{i}{2} + 1 \) if \( i \) is even, and \( 1 \) if \( i \) is odd. It is easy to prove that \( q'_i \leq F_i \) where \( F_i \) is the sequence defined by: \( F_{n+2} = \frac{p-1}{p} F_{n+1} + F_n \) with \( F_0 = 1 \) and \( F_1 = a_1 \).

In the other hand we have
\[
0 \leq q'_{i+2} - q'_i \leq \frac{p-1}{p^{A_{i+2} - A_{i+1} + 1}} F_{i+1}.
\]
So, for \( i \) even, we have
\[
0 \leq q'_{i+2} - q'_i \leq \frac{p-1}{p^{i+1} + 1} F_{i+1} \sim \frac{p-1}{p^{\frac{1}{2} + 1}} \left( \frac{1}{\sqrt{p}} \right)^n \Phi_{p_i}^{n-1}.
\]

**Theorem 2.3.** Suppose that for all \( i, a_i \in \{1, \ldots, p-1\} \), \( \alpha_{2j} = 1 \) for \( i = 2j \), and \( \alpha_{2j+1} = (j + 1)^2 \) for \( i = 2j + 1 \). Then the sequence \( [(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i] \) converges in \( \mathbb{R} \).
**Proof.** We have
\[ A_{2j} = 1 + 4 + \cdots + (2j)^2 = \frac{1}{6}j(j + 1)(2j + 1), A_{2j+1} = j + 1. \]
It is easy to prove that \( A_{2j+1} - A_{2j} = \frac{j+1}{6}(-2j^2 - j + 6) \). Then, we have
\[ q'_{2j+1} \leq p^{\frac{j+1}{6}}(-2j^2 - j + 6). \]
Finally, we make \( j \) tend to infinity.

More generally, it is possible to prove in the same way the following results.

**Theorem 2.4.** If for some \( \epsilon > 0 \) we have
\[ A_{i+2} - A_{i+1} > i(\epsilon + \log_p(\Phi_{p^{-1}})) \]
Then the sequence \((\frac{p_i}{q_i})_{i \in \mathbb{N}}\) does not converge in \( \mathbb{R} \).

**Theorem 2.5.** If we have \( A_{i+2} - A_{i+1} \leq \log_p(i) \) Then the sequence \((\frac{p_i}{q_i})_{i \in \mathbb{N}}\) converge in \( \mathbb{R} \).

**Conjecture 2.1.** We conjecture that: For \( \alpha_i = \lfloor \sqrt{i} \rfloor \), the sequence \((\frac{p_i}{q_i})_{i \in \mathbb{N}}\) does not converge in \( \mathbb{R} \). For \( \alpha_i = k \lfloor \log_p(i) \rfloor \), with \( k \geq 1 \), the sequence \((\frac{p_i}{q_i})_{i \in \mathbb{N}}\) converge in \( \mathbb{R} \).

In all the sequel, we denote \( \theta_\mathbb{R} \) the limit in \( \mathbb{R} \). In the proof of the transcendence theorem, we shall need the following lemma.

**Lemma 2.3.** Under the hypothesis of the previous theorem, we have:
\[ |q_n \theta_\mathbb{R} - p_n| \leq \frac{\prod_{j=0}^{j=n-1} p^\alpha_j}{q_{n-1}}. \]

**Proof.** From the Leibniz criterium for alternating series, we have
\[ \left| \theta_\mathbb{R} - \frac{p_n}{q_n} \right| \leq \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right|. \]

\[ \square \]

2.2. Convergence in \( \mathbb{Q}_p \).

**Theorem 2.6.** The sequence \([(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{n-1}, \alpha_{n-1}); a_n]\) converges in \( \mathbb{Q}_p \).
Proof. An easy recursion on \( n \) shows that \( \text{val}_p(q_n) = 0 \). Hence, the series

\[
\sum_{i=1}^{i=k} \frac{(-1)^i+1}{q_i q_{i-1}} \prod_{k=0}^{i-1} p^\alpha_k,
\]

have a limit in \( \mathbb{Q}_p \).

In all the sequel, we denote \( \theta_{\mathbb{Q}_p} \) the limit in \( \mathbb{Q}_p \). In the proof of the transcendence theorem, we shall need the following lemmas:

Lemma 2.4. Under the hypothesis of the previous proposition, we have:

\[
|q_k \theta_{\mathbb{Q}_p} - p_k|_p = \frac{1}{p^{\alpha_0 + \cdots + \alpha_k}}.
\]

Proof. Suppose \( k < n \), we have

\[
\frac{p_n}{q_n} - \frac{p_k}{q_k} = \sum_{i=k}^{i=n-1} \frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = \sum_{i=k}^{i=n-1} \frac{(-1)^{i+2} \prod_{j=0}^{i} p^\alpha_j}{q_i q_{i+1}}.
\]

Hence we have

\[
|\frac{p_n}{q_n} - \frac{p_k}{q_k}|_p = \frac{1}{p^{\alpha_0 + \cdots + \alpha_k}} \leq \frac{1}{p^{k+1}}.
\]

Then take the limit when \( n \) goes to infinity to get

\[
|\theta - \frac{p_k}{q_k}|_p = |q_k \theta - p_k|_p = \frac{1}{p^{\alpha_0 + \cdots + \alpha_k}}.
\]

3. Proof of main theorem

Suppose \( \theta \) is algebraic. Consider now the following linear forms with variable \( \mathbf{x} = (x_1, x_2, x_3) \) and algebraic coefficients.

\[
L_{1,\infty}(\mathbf{x}) = \theta_{\mathbb{R}} \cdot x_1 - x_3, \quad L_{2,\infty}(\mathbf{x}) = \theta_{\mathbb{R}} \cdot x_3 - x_2, \quad L_{3,\infty}(\mathbf{x}) = x_3,
\]

\[
L_{1,p}(\mathbf{x}) = \theta_{\mathbb{Q}_p} \cdot x_1 - x_3, \quad L_{2,p}(\mathbf{x}) = \theta_{\mathbb{Q}_p} \cdot x_3 - x_2, \quad L_{3,p}(\mathbf{x}) = x_1.
\]

Evaluating them on the triple \( \mathbf{x}_n = (q_n, p_{n-1}, p_n) \), with \( n = 4^k - 1 \). We get from Lemma 2.3 the inequality

\[
|L_{1,\infty}(\mathbf{x}_n)| < \frac{p^{\sum_{j=0}^{j=n-1} \alpha_j}}{q_{n-1}}.
\]
and from Theorem 1.3 and Lemma 2.3, the inequality
\[ |L_{2,\infty}(x_n)| \leq \frac{p^{\sum_{j=0}^{n-2} \alpha_j}}{q_{n-2}}. \]

For the p-adic forms, we have from Lemma 2.4
\[ |L_{1,p}(x_n)|_p = \frac{1}{p^{\sum_{j=0}^{n} \alpha_j}} \]
and
\[ |L_{2,p}(x_n)|_p = \frac{1}{p^{\sum_{j=0}^{n-1} \alpha_j}}. \]

From the hypothesis of the theorem, we have \( A < n \) then \( p_n < p^{n+1} \), for \( n = 4^k - 1 \) large enough.

\[ |x_n|^\delta \prod_{i=1}^{3} |L_{i,\infty}(x_n)| \prod_{i=1}^{3} |L_{i,p}(x_n)|_p < \frac{p_\delta}{p^{\alpha_{n-1}+\alpha_n} q_{n-2}} < \frac{p_\delta}{p^{2} q_{n-2}} < \frac{(p^{n+1})^\delta}{p^{2} + \frac{1}{2}} \]

converges to 0 for \( \delta < \frac{1}{2} \).

Schlickewei’s theorem confirms the existence of non-zero rational integers \( y_1, y_2, y_3 \), such that, for an infinite set of \( n \), we have
\[ y_1 q_n + y_2 p_{n-1} + y_3 p_n = 0, \quad \text{i.e.,} \quad y_1 + y_2 \frac{p_{n-1}}{q_n} + y_3 \frac{p_n}{q_n} = 0. \]

So, \( y_1 + y_2 \frac{p_{n-1}}{q_n} + y_3 \frac{p_n}{q_n} = 0. \)

Passing to the limit as \( n \to +\infty \) (in \( \mathbb{Q}_p \)), we obtain \( y_2 \theta_{\mathbb{Q}_p}^2 + y_3 \theta_{\mathbb{Q}_p} + y_1 = 0 \).
So, \( \theta_{\mathbb{Q}_p} \) is quadratic.

Passing to the limit as \( n \to +\infty \) (in \( \mathbb{R} \)), we obtain \( y_2 \theta_{\mathbb{R}}^2 + y_3 \theta_{\mathbb{R}} + y_1 = 0 \). So, \( \theta_{\mathbb{R}} \) is quadratic.

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REFERENCES


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